King Saud University
College of Sciences
Mathematics Department

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Solution of the second midterm exam ACTU-362-372 (25%) (two pages)

November 23, 2020 (two hours 10:15–12:15 AM)

Problem 1. (6 marks)

1. A fully continuous whole life insurance on (30) pays a benefit of 50,000 at the moment of death. The force of mortality is not constant for all ages and is given by:

$$\mu_{30+t} = 0.02 \text{ if } 0 \leq t < 20 \text{ and } \mu_{30+t} = \frac{1}{60-t} \text{ if } 20 \leq t < 60$$

   Calculate NSP the net single premium for this insurance when the force of interest is 4%.

2. On January 1, 2020, Sarah, aged 40, purchases a 5–payment, 10–year term insurance of 10^5
   (i) Death benefits are payable at the moment of death.
   (ii) Net premiums of 4000 are payable annually at the beginning of each year for 5 years.
   (iii) The effective interest rate is 5%
   (iv) ${}_0L$ is the loss random variable at time of issue.
   Calculate the value of ${}_0L$ if Sarah dies on June 30, 2024.

3. You are given: $\ell_{40} = 500$, $\ell_{41} = 475$, $i = 0.06$, and $\overline{A}_{41} = 0.54$. Deaths are uniformly distributed over each year of age. Calculate A_{40}.

Solution:

1. We can split the insurance into two components: a 20–year term insurance and a 20–year deferred insurance. In the initial 20–year period, The APV of the future benefits is

$$\text{NSP} = 50000(\bar{A}^{1}_{30:\overline{20}|} + {}_{20|}\bar{A}_{30}) = 50000(\bar{A}^{1}_{30:\overline{20}|} + {}_{20}E_{30}\ \bar{A}_{50})$$

   Moreover

$$\begin{aligned}\bar{A}^{1}_{30:\overline{20}|} &= \int_0^{20} e^{-\delta t}\ {}_tp_{30}\ \mu_{30+t}\ dt = \int_0^{20} 0.02e^{-(0.04+0.02)t}\ dt \\ &= \frac{0.02}{0.06}\left(1-e^{-(0.06)20}\right) = \frac{1}{3}\left(1-e^{-1.2}\right) = 0.23294\end{aligned}$$

   and

$${}_{20}E_{30}\ \bar{A}_{50} = e^{-(0.06)20}\int_0^{40} e^{-\delta t}\frac{1}{40}dt = \frac{e^{-1.2}}{40}\left(\frac{1-e^{-(0.04)40}}{0.04}\right) = 0.15024$$

   Thus NSP $= 50000\,(0.23294 + 0.15024) = \mathbf{19159}$.

2. The loss at issue of this insurance ${}_0L = 10^5 v^{\min(T_{40},10)} - 4000\ddot{a}_{\overline{\min(K_{40}+1,5)}|}$. If $T_{40} = 4.5$ then ${}_0L = 10^5 v^{T_{40}} - 4000\ddot{a}_{\overline{5}|} = 10^5 v^{T_{40}} - 4000\frac{1-v^5}{d} = \frac{10^5}{(1.05)^{4.5}} - 4000\frac{1-(1.05)^{-5}}{\frac{0.05}{1.05}} = \mathbf{62104}$.

3. We have

$$
\begin{aligned}
A_{40} &= vq_{40} + vp_{40}\ A_{41} = A_{40} = \frac{1}{1+i}\left(\left(1-\frac{\ell_{41}}{\ell_{40}}\right)+\frac{\ell_{41}}{\ell_{40}}\frac{\ln(1+i)}{i}\overline{A}_{41}\right) \\
&= \frac{1}{1.06}\left(\left(1-\frac{475}{500}\right)+\frac{475}{500}\ \frac{\ln(1.06)}{0.06}0.54\right) = \mathbf{0.51717}
\end{aligned}
$$

<u>Problem 2.</u> (6 marks)

1. You are given: (i) $A_x = 0.28$ (ii) $A_{x+20} = 0.40$ (iii) ${}_{20}E_x = 0.25$ (iv) $i = 0.05$. Calculate $a_{x:\overline{20}|}$.

2. Ismail is offered the choice of two continuous annuities:
   1. a life annuity, paying 50,000 per year
   2. a life annuity with a 5–year certain period, i.e., an annuity paying benefits for life, but with a minimum payout period of 5 years. The benefit amount for the second annuity is chosen so that the expected present value of the two annuities is the same.

   Determine the annual benefit amount on the second annuity given $\delta = 0.04$ and $\mu = 0.06$.

3. A special temporary 3–year life annuity–due on (40) pays $k \times 1000$ at the beginning of year k, $k = 1,\ 2,\ 3$. Mortality follows illustrative life table and $i = 0.06$. Find the actuarial present value of this annuity.

Solution.

1. We have

$$a_{x:\overline{20}|} = \ddot{a}_{x:\overline{20}|} - 1 + {}_{20}E_x$$

   where

$$\ddot{a}_{x:\overline{20}|} = \frac{1 - A_{x:\overline{20}|}}{d} = \frac{1-(A_x - {}_{20}E_x\ A_{x+20} + {}_{20}E_x)}{d} = \frac{1-(0.28-0.25\times 0.4+0.25)}{\frac{0.05}{1.05}} = 11.97$$

   hence $a_{x:\overline{20}|} = 11.97 - 1 + 0.25 = \mathbf{11.22}$.

2. The actuarial present value of the continuous whole life annuity of 1 is $\frac{1}{\mu+\delta} = \frac{1}{0.1} = 10$. The expected present value of the 5–year certain–and–life annuity is

$$\bar{a}_{\overline{x:\overline{5}|}} = \bar{a}_{\overline{5}|} + {}_{5|}\bar{a}_x$$

   and

$$
\begin{aligned}
\bar{a}_{\overline{5}|} &= \frac{1-e^{-5(0.04)}}{0.04} = 4.5317 \\
{}_{5|}\bar{a}_x &= \frac{e^{-5(0.04+0.06)}}{0.1} = 6.0653
\end{aligned}
$$

   thus

$$\bar{a}_{\overline{x:\overline{5}|}} = 4.5317 + 6.0653 = 10.597.$$

   The APVs of the two annuities are equal thus

$$b\bar{a}_{\overline{x:\overline{5}|}} = 50000 \times 10$$

   So the level benefit amount of the second annuity is $50000\left(\frac{10}{10.597}\right) = \mathbf{47183}$.

3. Denote by α the actuarial present value of this annuity, then we have

$$
\begin{aligned}
\alpha &= \sum_{k=0}^{1}\left(\sum_{i=0}^{k} v^i S_i\right) {}_kp_{40}\ q_{40+k} + \left(\sum_{i=0}^{2} v^i\ S_i\right)\ {}_2p_{40} \\
&= S_0\ q_{30} + (S_0 + S_1 v)\, p_{30}\ q_{31} + \left(S_0 + S_1 v + S_2 v^2\right)\ {}_2p_{30} \\
&= 1000\left(q_{40} + (1+2v)\, p_{40}\ q_{41} + \left(1 + 2v + 3v^2\right)\ p_{40}\ p_{41}\right) \\
&= 2.78 + \left(1 + \frac{2}{1.06}\right)\left(1 - \frac{2.78}{1000}\right) \times 2.98 \\
&\quad + 1000\left(1 + \frac{2}{1.06} + \frac{3}{(1.06)^2}\right)\left(1 - \frac{2.78}{1000}\right)\left(1 - \frac{2.98}{1000}\right) \\
&= \mathbf{5536.20}.
\end{aligned}
$$

Problem 3. (6 marks)

1. For a special fully discrete 20–year term insurance on (30):
   (i) The death benefit is 1000 during the first ten years and 3000 during the next ten years.
   (ii) The premium determined by the equivalence principle, is P for each of the **first ten** years and $3P$ for each of the **next ten** years.
   (iii)

x	$\ddot{a}_{x:\overline{10}\rvert}$	$\ddot{a}_{x:\overline{20}\rvert}$	$1000A^1_{x:\overline{10}\rvert}$
30	8.7201	15.0364	16.66
40	8.6602	–	32.61

   Calculate P.

2. For a special 2–year payment whole life insurance on (50):
   (i) Premiums of P are paid at the beginning of years 1 and 3.
   (ii) The death benefit is paid at the end of the year of death.
   (iii) There is a partial refund of premium feature: If (50) dies in either year 1 or year 3, the death benefit is $1000 + 0.8P$. Otherwise, the death benefit is 1000.
   (iv) Mortality follows the Illustrative Life Table and $i = 0.06$.
   Calculate P, using the equivalence principle.

Solution:

1. The APV of death benefit is

$$
\begin{aligned}
1000A^1_{30:\overline{10}\rvert} + 3000\ {}_{10\rvert}A^1_{30:\overline{10}\rvert} &= 1000A^1_{30:\overline{10}\rvert} + 3000\ {}_{10}E_{30}A^1_{40:\overline{10}\rvert} \\
&= 16.66 + 3 \times 32.61\ {}_{10}E_{30} = 16.66 + 97.83\ {}_{10}E_{30}
\end{aligned}
$$

   The APV of premiums can be written in different forms for example

$$3P\ddot{a}_{30:\overline{20}\rvert} - 2P\ddot{a}_{40:\overline{10}\rvert} = P\,(3 \times 15.0364 - 2 \times 8.7201) = 27.669P$$

   : By the equivalence principle, the premium is then

$$P = \frac{16.66 + 97.83\ {}_{10}E_{30}}{27.669}$$

   We still need to solve for ${}_{30}E_{10}$. To this end, we note that

$$\ddot{a}_{30:\overline{20}\rvert} = \ddot{a}_{30:\overline{10}\rvert} +\ {}_{10}E_{30}\ddot{a}_{40:\overline{10}\rvert} = 15.0364 = 8.7201 + 8.6602\ {}_{10}E_{30}$$

so that ${}_{10}E_{30} = 0.72935$. Therefore

$$P = \frac{16.66 + 97.83 \times 0.72935}{27.669} = \frac{88.012}{27.669} = \mathbf{3.1809}.$$

:

2. The actuarial present value of the insurance is

$$1000A_{50} + 0.8P\left(v\ q_{50} + v^3\ {}_{2|}q_{50}\right)$$

The actuarial present value of the premiums is $P\,(1 + v^2\ {}_2p_{50})$. We have

$${}_2p_{50} = \frac{\ell_{52}}{\ell_{50}} = \frac{8840770}{8950901} = 0.98770$$

and

$${}_{2|}q_{50} = {}_2p_{50}\ q_{52} = (0.98770)(0.00697) = 0.00688427.$$

thus

$$1 + v^2\ {}_2p_{50} = 1 + \frac{1}{(1.06)^2}\ \frac{\ell_{52}}{\ell_{50}} = 1 + \frac{1}{(1.06)^2}0.98770 = 1.8790495.$$

Hence

$$\begin{aligned} \mathrm{APV(FB)}_0 &= 1000\ A_{50} + 0.8P\left(v\ q_{50} + v^3\ {}_{2|}q_{50}\right) \\ &= 249.05 + 0.8P\left(\frac{0.00592}{1.06} + \frac{0.00688427}{(1.06)^3}\right) \\ &= 249.05 + 0.0090920572P \end{aligned}$$

By E.P. we get by equating the two,

$$1.8790495P = 249.05 + 0.0090920572P$$

Finally $P = \mathbf{133.18485}$.

Problem 4. (6 marks)

Consider a life aged (40) whose mortality follows De Moivre's law with a limiting age equals to 85.

1. Find the 20$^{\text{th}}$–percentile premium of a 10–year payment whole life insurance of 10000;
2. Find the 30$^{\text{th}}$–percentile premium of a 10–year payment whole life insurance of 10000;
3. Find the 30$^{\text{th}}$–percentile premium of a 10–year term life insurance of 10000
4. Find the 20$^{\text{th}}$–percentile premium of a 10–year term life insurance of 10000;
5. Find the 30$^{\text{th}}$–percentile premium of a 10–year endowment life insurance of 10000.
   Assume that the force of interest is $\delta = 0.05$ and that all the policies are **fully continuous**.

Solution:

1. We first find $t_{0.20}$. This is the solution to $F_{40}(t) = \frac{t}{45} = 0.20$, that is $t_{0.20} = 45 \times 0.2 = 9 < 10$. The 20$^{\text{th}}$–percentile premium is

$$P_{0.20} = \frac{10000}{\bar{s}_{\overline{t_{0.20}|}}} = \frac{10000 \times 0.05}{e^{0.05\times 9} - 1} = \mathbf{879.80}.$$

2. Clearly $t_{0.3} = 45 \times 0.3 = 13.5 > 10$, then 30^{th}–percentile premium is
$$P_{0.30} = \frac{10000e^{-0.05\times t_{0.30}}}{\bar{a}_{\overline{10}|}} = \frac{10000e^{-0.05\times 13.5}}{\frac{1-e^{-0.05\times 10}}{0.05}} = \mathbf{647.01}.$$

3. We have $t_{0.30} > 10$ thus $P_{0.30} = \mathbf{0}$.

4. We know from Q1. $t_{0.20} = 9 < 10$, so $P_{0.20}$ the 20^{th}–percentile premium is
$$P_{0.20} = \frac{10000}{\bar{s}_{\overline{9}|}} = \frac{10000 \times 0.05}{e^{0.05\times 9} - 1} = \mathbf{879.80}.$$

5. We have an endowment with $10 < t_{0.30}$, so the 30^{th}–percentile premium is
$$P_{0.30} = \frac{10000}{\bar{s}_{\overline{10}|}} = \frac{10000 \times 0.05}{e^{0.05\times 10} - 1} = \mathbf{770.75}.$$

Problem 5. (6 marks)

A fully discrete 4–year term insurance of 10^6 is sold to a life aged 30. Mortality are given by $q_{30+k} = \frac{3+k}{1000}$ for $k = 0, 1, 2, 3$. The level net premium is 4275.013

1. Calculate the net premium reserve **at time** 2 if $i = 0.04$ in all years.
2. For a fully discrete 10–year deferred whole life insurance of 1000 on (40), you are given: $v = 0.95$; $p_{48} = 0.98077$, $p_{49} = 0.98039$, $A_{50} = 0.35076$. The annual net premium of 23.40 is payable during the deferral period. Calculate ${}_8V$ the net premium reserve of this insurance immediately before the payment of the 10^{th} premium.
3. For a fully discrete whole life insurance of 4000 on (40), you are given $\mu_{40+t} = 0.02$ for $t < 10$ and $\mu_{40+t} = 0.04$ for $t \geq 10$ and the annual effective interest rate $i = 0.05$. The net level premium is 108.0804. Calculate ${}_{10}V$ and use recursion de calculate ${}_{10.5}V$.

Solution:

1. We shall use prospective method
$${}_2V = \text{APV(FB)}_2 - \text{APV(FP)}_2 = 10^6\ A^{1}_{32:\overline{2}|} - 4275.013\ \ddot{a}_{32:\overline{2}|}$$
Moreover, we have
$$A^{1}_{32:\overline{2}|} = v\ q_{32} + v^2\ p_{32}\ q_{33} = \frac{0.005}{1.04} + \frac{(0.995)(0.006)}{(1.04)^2} = 0.010327,$$
and
$$\ddot{a}_{32:\overline{2}|} = 1 + v\ p_{32} = 1 + \frac{0.995}{1.04} = 1.9567$$
hence ${}_2V = 10^6 \times 0.010327 - 4275.013 \times 1.9567 = \mathbf{1962.082.}$
Also by recursion we have
$$(\ {}_3V + P)\,(1+i) = b_4\ q_{33} +\ {}_4V\ p_{33} = b_4\ q_{33}$$
hence
$${}_3V = \frac{b_4\ q_{33}}{1+i} - P = \frac{1000 \times 6}{1.04} - 4275.013 = 1494.2$$
Similarly
$${}_2V = \frac{b_3\ q_{32} +\ {}_3V\ p_{32}}{1+i} - P = \frac{1000 \times 5 + 1494.2 \times 0.995}{1.04} - 4275.013 = 1962.2.$$

2. In this question we will use backward recursion since ${}_{10}V = 1000A_{50}$ (annual net premium is payable only during the deferral period).

$$\left({}_9V + P_9\right)(1+i) = \frac{{}_9V + P_9}{v} = b_{10}\; q_{49} + \;{}_{10}V\; p_{49} = \;{}_{10}V\; p_{49}\;(\text{since } b_{10} = 0)$$

so

$${}_9V = \left(v\;{}_{10}V\; p_{49} - P_9\right) = \left(v\; A_{50}\; p_{49} - P\right) = 0.95 \times 0.98039 \times 350.76 - 23.40 = 303.2875.$$

similarly

$${}_8V = \left(v\;{}_9V\; p_{48} - P\right) = 0.95 \times 303.2875 \times 0.98077 - 23.40 = \mathbf{259.18}.$$

3. Set $p_1 = e^{-0.02}$ and $p_2 = e^{-0.04}$, thus $q_1 = 1 - p_1 = 0.0198$ and $q_2 = 1 - p_2 = 0.03921$

By the **prospective method** we have

$$\begin{aligned} {}_{10}V &= 4000A_{50} - P\ddot{a}_{50} = 4000\frac{q_2}{q_2+i} - 108.0804\frac{1+i}{q_2+i} \\ &= 4000\frac{1-e^{-0.04}}{1.05-e^{-0.04}} - 108.0804\frac{1.05}{1.05-e^{-0.04}} = \mathbf{486.0168}. \end{aligned}$$

By the **retrospective method** we have

$$\begin{aligned} {}_{10}V &= \frac{P\ddot{a}_{45:\overline{10}|} - 4000A^{1}_{45:\overline{10}|}}{{}_{10}E_{45}} = \frac{108.0804\frac{1+i}{q_1+i}\left(1 - {}_{10}E_{45}\right) - 4000\frac{q_1}{q_1+i}\left(1 - {}_{10}E_{45}\right)}{{}_{10}E_{45}} \\ &= \left(108.0804\,(1+i) - 4000q_1\right)\frac{\left(1 - {}_{10}E_{45}\right)}{(q_1+i)\,{}_{10}E_{45}} \\ &= \left(108.0804 \times 1.05 - 4000\left(1 - e^{-0.02}\right)\right)\frac{\left(1 - (1.05)^{-10}\, e^{-0.2}\right)}{\left(1 - e^{-0.02} + 0.05\right)(1.05)^{-10}\, e^{-0.2}} \\ &= \left(108.0804 \times 1.05 - 4000\left(1 - e^{-0.02}\right)\right)\frac{(1.05)^{10}\, e^{0.2} - 1}{1.05 - e^{-0.02}} = \mathbf{485.9568}. \end{aligned}$$

Now, by **recursion** we can write

$$\left({}_{10}V + P\right)(1+i)^{0.5} = v^{0.5}4000\;{}_{0.5}q_{55} + \;{}_{10.5}V\;{}_{0.5}p_{55}$$

hence

$$\begin{aligned} {}_{10.5}V &= \frac{\left({}_{10}V + P\right)(1+i)^{0.5} - v^{0.5}4000\left(1 - {}_{0.5}p_{55}\right)}{{}_{0.5}p_{55}} \\ &= \frac{(486.0168 + 108.0804)\,(1.05)^{0.5} - (1.05)^{-0.5}\,4000\left(1 - e^{-0.04\times 0.5}\right)}{e^{-0.04\times 0.5}} = \mathbf{542.21}. \end{aligned}$$