King Saud University
College of Sciences
Mathematics Department

Academic Year (G) 2020–2021
Academic Year (H) 1442
Bachelor AFM: M. Eddahbi

October 4, 2020 from 9 to 11 AM, ACTU-362-372 (two pages + formula sheet)

Midterm Exam I, Actuarial Mathematical Models I

Exercise 1

The survival function $S_0(t)$ of the age-at death T_0 is given by

$$S_0(t) = \frac{1}{(1+t)^2} \text{ for } \quad t \geq 0,$$

1. *Find $F_0(t)$ the c.d.f of T_0*
2. *Find $f_0(t)$ the p.d.f of T_0*
3. *Find $S_x(t)$ the survival function of the future life time T_x of an age x.*
4. *Calculate ${}_2p_{10}$.*
5. *Calculate ${}_{10|5}q_{20}$.*

Solution:

1. The c.d.f. $F_0(t)$ of T_0 is given by $F_0(t) = 1 - S_0(t) = 1 - \frac{1}{(1+t)^2}$ for all $t \geq 0$.
2. The p.d.f. $f_0(t)$ of T_0 is given by $f_0(t) = F_0'(t) = \left(1 - \frac{1}{(1+t)^2}\right)' = \frac{2}{(1+t)^3}$ for all $t \geq 0$.
3. The survival function $S_x(t)$ of future life time T_x is given in terms of $S_0(t)$ by: $\frac{2}{(1+t)^3}$

$$S_x(t) = \frac{S_0(x+t)}{S_0(x)} = \frac{\frac{1}{(1+x+t)^2}}{\frac{1}{(1+x)}} = \frac{(1+x)^2}{(1+x+t)^2}.$$

4. We know that ${}_tp_x = S_x(t)$, then ${}_2p_{10} = S_{10}(2) = \frac{(1+10)^2}{(1+10+2)^2} = \frac{(11)^2}{(13)^2} = \mathbf{0.71598}$.
5. We know that ${}_{m|n}q_x = {}_{m+n}q_x - {}_mq_x = {}_mp_x - {}_{m+n}p_x$, then

$$\begin{aligned} {}_{10|5}q_{20} &= {}_{10}p_{20} - {}_{15}p_{20} = S_{20}(10) - S_{20}(15) \\ &= \frac{(1+20)^2}{(1+20+10)^2} - \frac{(1+20)^2}{(1+20+15)^2} = \mathbf{0.11862}. \end{aligned}$$

Exercise 2

1. *You are given the following mortality table:*

x	20	21	22	23	24
q_x				0.0700	
ℓ_x	$30,000$				$23,900$
d_x	$1,200$				

Determine the probability that a life age 21 *will die within two years.*

2. ***CANCELED*** *You are given* ${}_{k|}q_x = 0.01 + \frac{k}{100}$ *for* $k = 0,\ 1,\ \ldots,\ 9$. *Calculate* ${}_4p_{x+6}$

3. *Find* ${}_tp_x$ *when the Mortality follows Gompertz's law* ($\mu_x = Bc^x$)*. You are given that* ${}_5p_{60} = 0.95$ *and* ${}_{10}p_{60} = 0.87$*. Determine* ${}_{30}p_{60}$*.*

Solution:

1. The required probability is ${}_2q_{21} = 1 - \frac{\ell_{23}}{\ell_{21}}$. We need ℓ_{21} and ℓ_{23} which will be backed up as follows

$$\ell_{21} = \ell_{20} - d_{20} = 30000 - 1200 = 28800$$

now we shall use $q_x = \frac{\ell_x - \ell_{x+1}}{\ell_x}$ or equivalently $\ell_x = \frac{\ell_{x+1}}{1-q_x}$, thus $\ell_{23} = \frac{\ell_{24}}{1-q_{23}} = \frac{23900}{1-0.07} = 25698.92$. So we get

$${}_2q_{21} = 1 - \frac{25698.92}{28800} = \mathbf{0.1076764}.$$

2. **CANCELED** By definition of ${}_tp_x$ we can write

$${}_4p_{x+6} = \frac{\ell_{x+10}}{\ell_{x+6}} \text{ and } {}_{k|}q_x = \frac{\ell_{x+k} - \ell_{x+1+k}}{\ell_x} \iff \ell_{x+k} - \ell_{x+1+k} = {}_{k|}q_x\ \ell_x$$

hence

$$\begin{aligned} \ell_x - \ell_{x+10} &= \sum_{k=0}^{9} (\ell_{x+k} - \ell_{x+1+k}) = \ell_x \sum_{k=0}^{9} {}_{k|}q_x \\ \ell_x - \ell_{x+6} &= \sum_{k=0}^{5} (\ell_{x+k} - \ell_{x+1+k}) = \ell_x \sum_{k=0}^{5} {}_{k|}q_x \end{aligned}$$

thus

$$\ell_{x+10} = \ell_x \left(1 - \sum_{k=0}^{9} {}_{k|}q_x\right) \text{ and } \ell_{x+6} = \ell_x \left(1 - \sum_{k=0}^{5} {}_{k|}q_x\right)$$

therefore

$${}_4p_{x+6} = \frac{\ell_{x+10}}{\ell_{x+6}} = \frac{1 - \sum_{k=0}^{9} {}_{k|}q_x}{1 - \sum_{k=0}^{5} {}_{k|}q_x} = \frac{1 - \sum_{k=0}^{9}(0.01 + 0.01k)}{1 - \sum_{k=0}^{5}(0.01 + 0.01k)} = \frac{0.55}{0.79} = \mathbf{0.6962}.$$

3. By definition of ${}_tp_x$ we can write

$$ {}_tp_x = e^{-\int_x^{x+t}\mu_s ds} = e^{-B\int_x^{x+t}c^s ds} = e^{-B\int_0^t c^{x+s}ds} = e^{-\frac{Bc^x}{\ln(c)}(c^t-1)} $$

Take the log for $x = 60$, $t = 5$ and $t = 10$ in the previous formula we get,

$$ \frac{Bc^{60}\left(c^5-1\right)}{\ln(c)} = -\ln(0.95) \quad \text{and} \quad \frac{Bc^{60}\left(c^{10}-1\right)}{\ln(c)} = -\ln(0.87) $$

Dividing the first into the second,

$$ \begin{aligned} c^5+1 &= \frac{\ln(0.87)}{\ln(0.95)} = 2.715015 \\ c &= \sqrt[5]{1.715015} = 1.11392 \\ c^{30}-1 &= 24.4453 \\ \frac{Bc^{60}\left(c^{30}-1\right)}{\ln(c)} &= \left(\frac{Bc^{60}\left(c^5-1\right)}{\ln(c)}\right)\left(\frac{c^{30}-1}{c^5-1}\right) \\ &= -\ln({}_5p_{60})\left(\frac{c^{30}-1}{c^5-1}\right) \\ &= (-\ln 0.95)\left(\frac{c^{30}-1}{c^5-1}\right) = 1.753641 \\ {}_{30}p_{60} &= e^{-1.753641} = 0.1731 \end{aligned} $$

Exercise 3

1. *Assume that the life table function is given by* $\ell_x = 10000e^{-0.05x}$, $x \geq 0$. *find* ${}_{3|10}q_{25}$
2. *Assume that the life table function at some ages is given as follows: (i)* $\ell_{40} = 94,000$, *(ii)* $\ell_{41} = 93,000$, *(ii)* $\ell_{42} = 92,000$
   Assuming Constant Force of Mortality between integral ages, find ${}_{1.3}q_{40.5}$.
3. *The force of mortality at some ages of a certain mortality table are given: (i)* $\mu_{60.5} = 0.02$, *(ii)* $\mu_{61.5} = 0.04$ *and (iii)* $\mu_{62.5} = 0.06$. *Under* ***UDD*** *assumption, calculate the probability that a person age* 60.5 *will die within two years.*

Solution:

1. We have

$$ \begin{aligned} {}_{3|10}q_{25} &= {}_{13}q_{25} - {}_3q_{25} = {}_3p_{25} - {}_{13}p_{25} = \frac{\ell_{28}}{\ell_{25}} - \frac{\ell_{38}}{\ell_{25}} \\ &= \frac{\ell_{28}-\ell_{38}}{\ell_{25}} = \frac{e^{-0.05\times 28} - e^{-0.05\times 38}}{e^{-0.05\times 25}} \\ &= e^{-0.05\times 3} - e^{-0.05\times 13} = \mathbf{0.3386} \end{aligned} $$

2. We can write under **CFM** assumption

$$ \begin{aligned} {}_{1.3}p_{40.5} &= \frac{{}_{1.8}p_{40}}{{}_{0.5}p_{40}} = \frac{p_{40}\times {}_{0.8}p_{41}}{{}_{0.5}p_{40}} = \frac{p_{40}\left(p_{41}\right)^{0.8}}{\left(p_{40}\right)^{0.5}} = \left(p_{40}\right)^{0.5}\left(p_{41}\right)^{0.8} \\ &= \left(\frac{\ell_{41}}{\ell_{40}}\right)^{0.5}\left(\frac{\ell_{42}}{\ell_{41}}\right)^{0.8} = \left(\frac{93}{94}\right)^{0.5}\left(\frac{92}{93}\right)^{0.8} = 0.9861. \end{aligned} $$

thus

$$ {}_{1.3}q_{40.5} = 1 - {}_{1.3}p_{40.5} = 1 - 0.9861 = \mathbf{0.0139}. $$

3. The probability that a person age 60.5 will die within two years is given by

$$ P\left(T_{60.5} \leq 2\right) = {}_{2}q_{60.5} = 1 - {}_{2}p_{60.5}. $$

First calculate ${}_{2}p_{60.5}$. We have

$$ \begin{aligned} {}_{2}p_{60.5} &= \frac{{}_{2.5}p_{60}}{{}_{0.5}p_{60}} = \frac{{}_{2}p_{60} \times {}_{0.5}p_{62}}{{}_{0.5}p_{60}} = \frac{p_{60} \times p_{61} \times {}_{0.5}p_{62}}{{}_{0.5}p_{60}} \\ &= \frac{\left(1 - q_{60}\right)\left(1 - q_{61}\right)\left(1 - 0.5q_{62}\right)}{\left(1 - 0.5q_{60}\right).} \end{aligned} $$

Remember that under **UDD** assumption we have

$$ \mu_{x+r} = \frac{q_x}{1 - rq_x} \text{ for all integer } x \text{ and } 0 < r < 1 $$

Therefore $q_x = \frac{\mu_{x+r}}{1+r\mu_{x+r}}$ and then i) $q_{60} = \frac{0.02}{1+0.5\times 0.02} = 0.0198$, ii) $q_{61} = \frac{0.04}{1+0.5\times 0.04} = 0.0392$ and iii) $q_{62} = \frac{0.06}{1+0.5\times 0.06} = 0.0582$, consequently

$$ {}_{2}p_{60.5} = \frac{\left(1 - 0.0198\right)\left(1 - 0.0392\right)\left(1 - 0.5 \times 0.0582\right)}{\left(1 - 0.5 \times 0.0198\right).} = 0.92351 $$

Finally, the required probability ${}_{2}q_{60.5} = 1 - 0.92351 = \mathbf{0.07649}$.

Exercise 4

1. *Ismail at age* 30*, wants to buy a* 3*–year endowment insurance, with a* 40,000 *SAR benefit payable at the end of the year of death. Determine the net single premium for this insurance assuming that Ismail is subject to a constant force of mortality,* $\mu_x = 0.08$ *and a constant force of interest* $\delta = 0.04$.

2. *A Woman, age* 36*, purchases a* 3*–year term insurance with benefits payable at the end of the year of death. The benefit is given by the formula:* $b_k = 5000 \times (1.04)^k$ *for* $t = 1, 2, 3$ *Mortality is described in the following table:*

Age	35	36	37	38	39	40
ℓ_x	60,000	50,000	47,500	45,000	42,500	40,000

Calculate the net single premium for this insurance when $i = 0.05$.

3. *You are given the following excerpt from a **TWO year** select–and–ultimate mortality table:*

$[x]$	$\ell_{[x]}$	$\ell_{[x]+1}$	ℓ_{x+2}	$x+2$
30	10,000	9,000	8,000	32
31	9,500	8,500	7,500	33
32	9,200	8,100	7,000	34
33	8,700	7,500	6,300	35
34	8,000	6,600	5,200	36

You are also given that a 4*–year pure endowment for* 250,000 *SAR issued at age* 30 *to a life selected at age* 30 *has a net single premium of* 50,000 *SAR Assume a constant rate of interest. Determine the net single premium for a* 3*-year pure endowment for* 250,000 *SAR issued at age* 31 *to a life selected at age* 31.

Solution:

1. The net single premium is $40000 A_{30:\overline{3}|}$ where

$$\begin{aligned} A_{30:\overline{3}|} &= A^{\ 1}_{30:\overline{3}|} + {}_3E_{30} \ = vq_{30} + v^2 p_{30} q_{31} + v^3 \ {}_2p_{30} q_{32} + v^3 \ {}_3p_{30} \\ &= vq_{30} + v^2 p_{30} q_{31} + v^3 \ {}_2p_{30} \left(q_{32} + p_{32} \right) = vq_{30} + v^2 p_{30} q_{31} + v^3 \ {}_2p_{30} \\ &= e^{-0.04} \left(1 - e^{-0.08} \right) + e^{-0.08} \left(e^{-0.08} \right) \left(1 - e^{-0.08} \right) + e^{-0.12} e^{-0.16} \\ &= 0.07387 + 0.65516 + 0.75578 = 1.4848 \end{aligned}$$

The answer is therefore $40000(1.4848) =$ **59392**.

2. The net single premium NSP is given by:

$$\begin{aligned} \text{NSP} &= vb_1 \ q_{36} + v^2 b_2 \ p_{36} \ q_{37} + v^3 b_3 \ {}_2p_{36} \ q_{38} \\ &= vb_1 \frac{\ell_{36} - \ell_{37}}{\ell_{36}} + v^2 b_2 \ \frac{\ell_{37}}{\ell_{36}} \frac{\ell_{37} - \ell_{38}}{\ell_{37}} + v^3 b_3 \frac{\ell_{38}}{\ell_{36}} \frac{\ell_{38} - \ell_{39}}{\ell_{38}} \\ &= \frac{1}{\ell_{36}} \left(vb_1 \left(\ell_{36} - \ell_{37} \right) + v^2 b_2 \left(\ell_{37} - \ell_{38} \right) + v^3 b_3 \left(\ell_{38} - \ell_{39} \right) \right) \\ &= \frac{5000}{\ell_{36}} \left(\frac{1.04}{1.05} \left(\ell_{36} - \ell_{37} \right) + \left(\frac{1.04}{1.05} \right)^2 \left(\ell_{37} - \ell_{38} \right) + \left(\frac{1.04}{1.05} \right)^3 \left(\ell_{38} - \ell_{39} \right) \right) \\ &= \frac{5000}{50000} \left(\frac{1.04}{1.05} (50000 - 47500) + \left(\frac{1.04}{1.05} \right)^2 (47500 - 45000) + \left(\frac{1.04}{1.05} \right)^3 (45000 - 42500) \right) \\ &= \mathbf{735.8} \end{aligned}$$

3. The NSP of this policy is given by $250000 A_{[31]:\overline{3}|}^{\ \ \ \ 1} = 250000 \ {}_3E_{[31]}$ where

$$A_{[31]:\overline{3}|}^{\ \ \ \ 1} = {}_3E_{[31]} = v^3 \ {}_3p_{[31]} = v^3 \ {}_3p_{[31]} = v^3 \frac{\ell_{[31]+3}}{\ell_{[31]}}.$$

Now, let us back out the discount factor. We have

$$\begin{aligned} 50000 &= 250000 \ v^4 \frac{\ell_{[30]+4}}{\ell_{[30]}} \iff 1 = 5v^4 \frac{5200}{10000} = 0.52 \times 5v^4 \\ & \quad 0.52 \times 5 \end{aligned}$$

which gives $v = \left(\frac{1}{2.6} \right)^{0.25} = 0.78751$.
Therefore the Net Single Premium for the 3–year pure endowment is .

$$250000 \ {}_3E_{[31]} = 250000 \left(0.78751 \right)^3 \left(\frac{7000}{9500} \right) = \mathbf{89967}.$$

Exercise 5

1. *You are given: (i) The benefit of* $120,000$ *on a ten–year endowment insurance is payable at the moment of death, or at the end of* 10 *years if* (x) *survives* 10 *years. (ii)* $\mu_{x+t} = 0.02$ *for* $t > 0$ *(iii)* $\delta = 0.04$*. Calculate the expected present value.*

2. *The expected present value of an* n*–year term insurance paying* $10,000$ *at the moment of death to* (x) *is* 36125*. You are given: (i)* $\mu_{x+t} = 0.008,\ t > 0$ *(ii)* $\delta = 0.032$*. Determine* n

3. *A* 5*–year deferred whole life insurance of* $200,000$ *on* (x) *is payable at the moment of death. You are given that*

$$\mu_{x+t} = 0.02 \text{ for all } t \geq 0 \quad \text{and } \delta_t = 0.06 \text{ for all } t \geq 0$$

Calculate the expected present value (net single premium) of this insurance.

Solution:

1. The APV is given by $120000\bar{A}_{x:\overline{10}|}$ and we know that

$$\bar{A}_{x:\overline{n}|} = \bar{A}^{1}_{x:\overline{n}|} + {}_{n}E_x = \frac{\mu}{\mu+\delta}\left(1 - e^{-n(\mu+\delta)}\right) + e^{-n(\mu+\delta)}$$

for $n = 10$, $\mu = 0.02$ and $\delta = 0.04$

$$\bar{A}_{x:\overline{10}|} = \frac{0.02}{0.02+0.04}\left(1 - e^{-10(0.06)}\right) + e^{-10(0.06)} = \mathbf{0.69921}.$$

Finally APV is given by $120000\bar{A}_{x:\overline{10}|} = 120000 \times 0.69921 = \mathbf{83905}$.

2. We know that $\bar{A}^{1}_{x:\overline{n}|} = \frac{\mu}{\mu+\delta}\left(1 - e^{-n(\mu+\delta)}\right) = \frac{0.008}{0.04}\left(1 - e^{-0.04n}\right) = 0.8e^{-0.04n} = 0.36125$, which gives $n = 19.876 \simeq 20$.

3. The APV of the insurance is given by $200000\,{}_{5|}\bar{A}_x$ where

$${}_{5|}\bar{A}_x = \int_5^\infty e^{-\delta t}\ {}_tp_x\ \mu_{x+t}\ dt = \int_5^\infty e^{-\delta t}\ e^{-\mu t}\mu\ dt = \frac{\mu}{\mu+\delta}e^{-5(\mu+\delta)} = \frac{0.02}{0.08}e^{-5\times 0.08} = 0.16758$$

so the APV is $200000 \times 0.16758 = \mathbf{33516}$.