King Saud University
College of Sciences
Mathematics Department

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Solution of the First midterm exam ACTU. 362 + ACTU. 372

Spring 2020, March 2 (two hours 3–5 PM)

Use ballpoint or ink–jet pens and keep three digits after dot

__Problem 1.__ **(5 marks)**

1. Express the probabilities associated with the following events in actuarial notation.
   (a) **(1 mark)** A person age 10 now survives to age 35.
   (b) **(1 mark)** A new born infant dies no later than age 40.
   (c) **(1 mark)** A person age 40 now survives to age 50 but dies before attaining age 65.
2. **(3 marks)** Assuming that $S_0(t) = e^{-0.05t}$ for $t \geq 0$. Calculate the probabilities of **questions (a), (b) and (c)**

Solution:

1. According to the actuarial notations we have
   (a) The probability that a person age $x = 10$ now survives to age $35 = 10+25$ can be expressed as
   $$S_{10}(25) = P\left(T_{10} \geq 25\right) = {}_{\mathbf{25}}\mathbf{p}_{\mathbf{10}}.$$
   (b) The probability that a new born infant ($x = 0$) dies no later than age $t = 40$ can be expressed as
   $$P\left(T_0 \leq 40\right) = {}_{\mathbf{40}}\mathbf{q}_{\mathbf{0}}.$$
   (c) The probability that a person age 40 now survives to age 50 but dies before attaining age 65 can be expressed as ($x = 40$, $t = 50 - 40 = 10$, and $u = 65 - 50 = 15$)
   $$P\left(10 \leq T_{40} \leq 10 + 15\right) = {}_{\mathbf{10|15}}\mathbf{q}_{\mathbf{40}}.$$
2. Given $S_0(t) = e^{-0.05t}$ for $t \geq 0$, we have
   $${}_{25}p_{10} = \frac{S_0(35)}{S_0(10)} = \frac{e^{-0.05\times 35}}{e^{-0.05\times 10}} = e^{-0.05\times 25} = \mathbf{0.286}.$$
   and
   $${}_{40}q_0 = 1 - {}_{40}p_0 = 1 - S_0(40) = 1 - e^{-0.05\times 40} = \mathbf{0.865}.$$
   and
   $$\begin{aligned} {}_{10|15}q_{40} &= {}_{10}p_{40} - {}_{25}p_{40} = \frac{e^{-0.05\times 50}}{e^{-0.05\times 40}} - \frac{e^{-0.05\times 65}}{e^{-0.05\times 40}} \\ &= e^{-0.05\times 10} - e^{-0.05\times 25} = \mathbf{0.320}. \end{aligned}$$

Problem 2. (6 marks)

The force of mortality μ_x is given by

$$\mu_x = \frac{2}{100-x} \text{ for } \quad 0 \le x < 100,$$

1. **(2 mark)** Find $S_{20}(t)$ and $f_{20}(t)$ for $0 \le t < 80$.
2. **(2 mark)** Compute ${}_{10}q_{20}$ and the probability that an **aged (20) dies** within 30 years.
3. **(2 marks)** Assume that $f_0(t) = \frac{20-t}{200}$ for $0 \le t < 20$. Find μ_{10}

Solution:

1. We know that for $0 \le u \le t < 80$, $0 \le 20+u \le 20+t < 100$, therefore

$$\begin{aligned} S_{20}(t) &= e^{-\int_0^t \mu_{20+u}du} = e^{-\int_0^t \frac{2}{80-u}du} = e^{2\int_0^t \frac{d(80-u)}{80-u}} \\ &= e^{2[\ln(80-t)-\ln(80)]} = e^{2\ln\left(1-\frac{t}{80}\right)} = \left(\mathbf{1}-\frac{\mathbf{t}}{\mathbf{80}}\right)^2, \text{ for } 0 \le t < 80. \end{aligned}$$

and

$$\begin{aligned} f_{20}(t) &= -S'_{20}(t) = -\left(\left(1-\frac{t}{80}\right)^2\right)' \\ &= \frac{\mathbf{1}}{\mathbf{40}}\left(\mathbf{1}-\frac{\mathbf{t}}{\mathbf{80}}\right) \quad \text{for} \quad 0 \le t < 80. \end{aligned}$$

2. We have

$$\begin{aligned} {}_{10}q_{20} &= 1 - {}_{10}p_{20} = 1 - S_{20}(10) = 1-\left(1-\frac{10}{80}\right)^2 \\ &= 1-\left(1-\frac{1}{8}\right)^2 = 1-\left(\frac{7}{8}\right)^2 = \frac{15}{64} = \mathbf{0.234}. \end{aligned}$$

and the probability that an **aged (20) dies** within 30 years is given by

$$\begin{aligned} P\left(0 \le T_{20} \le 30\right) &= S_{20}(0) - S_{20}(30) \\ &= 1-\left(1-\frac{3}{8}\right)^2 = \frac{39}{64} = \mathbf{0.6094}. \end{aligned}$$

3. We know that $f_0(t) = S_0(t)\mu_t$. So we need to calculate first $S_0(t)$. For $0 \le t < 20$, we have

$$\begin{aligned} S_0(t) &= 1 - F_0(t) = 1 - \int_0^t f_0(u)du = 1 - \int_0^t \frac{20-u}{200}du \\ &= 1 - \int_0^t \frac{20-u}{200}du = 1 + \left[\frac{(20-u)^2}{400}\right]_0^t \\ &= \frac{(20-t)^2}{400}. \end{aligned}$$

and then

$$\mu_t = \frac{f_0(t)}{S_0(t)} = \frac{\frac{20-t}{200}}{\frac{(20-t)^2}{400}} = \frac{2}{20-t}, \; 0 \le t < 20, \quad \text{hence} \quad \mu_{10} = \frac{2}{10} = \frac{\mathbf{1}}{\mathbf{5}}.$$

<u>Problem 3.</u> (6 marks)

1. **(2 marks)** Given $\ell_{[45]} = 1000$, ${}_5q_{[45]} = 0.04$ and ${}_5q_{[45]+5} = 0.05$, calculate $\ell_{[45]+10}$.
2. **(2 marks)** Now, assume that $q_{[45]+k} = 0.05$ for $k \geq 0$. Calculate $e_{[45]}$.
3. **(2 marks)** Assume that the future lifetime T_{20} is subject to force of mortality

$$\mu_x = \frac{1}{100-x} \quad \text{for} \quad 0 \leq x < 100.$$

a. Calculate e_{20}, b. Calculate $e_{20:\overline{50}|}$.

Solution:

1. We know ${}_{10}p_{[45]} = \dfrac{\ell_{[45]+10}}{\ell_{[45]}}$, then

$$\begin{aligned} \ell_{[45]+10} &= \ell_{[45]}\ {}_{10}p_{[45]} = \ell_{[45]}\ {}_5p_{[45]}\ {}_5p_{[45]+5} = \ell_{[45]}\left(1 - {}_5q_{[45]}\right)\left(1 - {}_5q_{[45]+5}\right) \\ &= 1000\,(1-0.04)\,(1-0.05) = \mathbf{912}. \end{aligned}$$

2. We know that $e_{[45]} = \sum_{k=1}^{\infty} {}_kp_{[45]} = \sum_{k=1}^{\infty} (0.95)^k = \frac{0.95}{1-0.95} = \mathbf{19}$.

3. a. We know $e_{20} = \mathring{e}_x - 0.5$,

$$\mathring{e}_{20} = \int_0^{100-20} {}_tp_{20}dt \quad \text{and} \quad {}_tp_{20} = e^{\int_{20}^{20+t} \frac{-du}{100-u}} = e^{[\ln(80-t)-\ln(80)]} = 1 - \frac{t}{80}$$

thence

$$\mathring{e}_{20} = \int_0^{80} \left(1 - \frac{t}{80}\right) dt = 40, \text{ thus } \ e_{20} = 40 - 0.5 = \mathbf{39.5}.$$

b. and

$$\mathring{e}_{20:\overline{50}|} = \int_0^{50} {}_tp_{20}dt = \int_0^{50} \left(1 - \frac{t}{80}\right) dt = \int_0^{50} \left(1 - \frac{t}{80}\right) dt = \frac{275}{8} = 34.375$$

Therefore using the formula

$$\mathring{e}_{20:\overline{50}|} = e_{20:\overline{50}|} + \frac{1}{2}\ {}_{50}q_{20} = e_{20:\overline{50}|} + \frac{1}{2}\left(1 - {}_{50}p_{20}\right)$$

we get

$$e_{20:\overline{50}|} = \mathring{e}_{20:\overline{50}|} - \frac{1}{2}\left(1 - {}_{50}p_{20}\right) = 34.375 - \frac{1}{2}\frac{50}{80} = \mathbf{34.063}$$

Problem 4. (6 marks)

1. **(2 marks)** Assume that the life table function is given by $\ell_x = 10000e^{-0.05x}$, $x \geq 0$. find ${}_{3|10}q_{25}$
2. Assume that the life table function at some ages is given as follows: (i) $\ell_{40} = 94,000$, (ii) $\ell_{41} = 93,000$, (ii) $\ell_{42} = 92,000$
   (2 marks) Assuming Constant Force of Mortality between integral ages, find ${}_{1.3}q_{40.5}$.
3. **(2 marks)** The force of mortality at some ages of a certain mortality table are given: (i) $\mu_{60.5} = 0.02$, (ii) $\mu_{61.5} = 0.04$ and (iii) $\mu_{62.5} = 0.06$. Under **UDD** assumption, calculate the probability that a person age 60.5 will die within two years.

Solution:

1. We have

$$\begin{aligned} {}_{3|10}q_{25} &= {}_{13}q_{25} - {}_{3}q_{25} = {}_{3}p_{25} - {}_{13}p_{25} = \frac{\ell_{28}}{\ell_{25}} - \frac{\ell_{38}}{\ell_{25}} \\ &= \frac{\ell_{28} - \ell_{38}}{\ell_{25}} = \frac{e^{-0.05\times 28} - e^{-0.05\times 38}}{e^{-0.05\times 25}} \\ &= e^{-0.05\times 3} - e^{-0.05\times 13} = \mathbf{0.3386} \end{aligned}$$

2. We can write under **UDD** assumption

$$\begin{aligned} {}_{1.3}p_{40.5} &= \frac{{}_{1.8}p_{40}}{{}_{0.5}p_{40}} = \frac{p_{40} \times {}_{0.8}p_{41}}{{}_{0.5}p_{40}} = \frac{p_{40}\left(p_{41}\right)^{0.8}}{\left(p_{40}\right)^{0.5}} = \left(p_{40}\right)^{0.5}\left(p_{41}\right)^{0.8} \\ &= \left(\frac{\ell_{41}}{\ell_{40}}\right)^{0.5}\left(\frac{\ell_{42}}{\ell_{41}}\right)^{0.8} = \left(\frac{93}{94}\right)^{0.5}\left(\frac{92}{93}\right)^{0.8} = 0.9861. \end{aligned}$$

   thus

$${}_{1.3}q_{40.5} = 1 - {}_{1.3}p_{40.5} = 1 - 0.9861 = \mathbf{0.0139}.$$

3. The probability that a person age 60.5 will die within two years is given by

$$P\left(T_{60.5} \leq 2\right) = {}_{2}q_{60.5} = 1 - {}_{2}p_{60.5}.$$

   First calculate ${}_{2}p_{60.5}$. We have

$$\begin{aligned} {}_{2}p_{60.5} &= \frac{{}_{2.5}p_{60}}{{}_{0.5}p_{60}} = \frac{{}_{2}p_{60} \times {}_{0.5}p_{62}}{{}_{0.5}p_{60}} = \frac{p_{60} \times p_{61} \times {}_{0.5}p_{62}}{{}_{0.5}p_{60}} \\ &= \frac{\left(1 - q_{60}\right)\left(1 - q_{61}\right)\left(1 - 0.5q_{62}\right)}{\left(1 - 0.5q_{60}\right).} \end{aligned}$$

   Remember that under **UDD** assumption we have

$$\mu_{x+r} = \frac{q_x}{1 - rq_x} \text{ for all integer } x \text{ and } 0 < r < 1$$

   Therefore $q_x = \frac{\mu_{x+r}}{1+r\mu_{x+r}}$ and then i) $q_{60} = \frac{0.02}{1+0.5\times 0.02} = 0.0198$, ii) $q_{61} = \frac{0.04}{1+0.5\times 0.04} = 0.0392$ and iii) $q_{62} = \frac{0.06}{1+0.5\times 0.06} = 0.0582$, consequently

$${}_{2}p_{60.5} = \frac{\left(1 - 0.0198\right)\left(1 - 0.0392\right)\left(1 - 0.5 \times 0.0582\right)}{\left(1 - 0.5 \times 0.0198\right).} = 0.92351$$

   Finally, the required probability ${}_{2}q_{60.5} = 1 - 0.92351 = \mathbf{0.07649}$.

Problem 5. (6 marks)

1. **(3 marks)** A 20–year endowment insurance on (45) pays 1000 at the moment of death if death occurs within 10 years, 500 at the moment of death if it occurs after 10 years, and 500 at the end of 20 years if the insured is alive. You are given:
   (i) Mortality follows the **Illustrative Life Table** and $\mathbf{i = 6}\%$.
   (ii) Deaths are **uniformly distributed** between integral ages.
   Calculate the net single premium for the endowment insurance.

2. **(3 marks)** For a special 3–year endowment insurance on (x), you are given the following death probabilities and death benefits, payable at the end of the year of death:

year k	b_k	q_{x+k-1}
1	30	0.02
2	35	0.04
3	40	**NA**

   The survival benefit is equal to b_3. Calculate the expected present value of this insurance for $i = 4\%$.

Solution:

1. The net single premium **NSP** is given by APV of future benefits

$$\begin{aligned}\mathbf{NSP} &= 500\bar{A}_{45:\overline{20|}} + 500\bar{A}^{\,1}_{45:\overline{10|}} = 500\left(\bar{A}_{45} - {}_{20}E_{45}\bar{A}_{65} + {}_{20}E_{45}\right) + 500\left(\bar{A}_{45} - {}_{10}E_{45}\bar{A}_{55}\right) \\ &= 1000\bar{A}_{45} - 500\left({}_{20}E_{45}\bar{A}_{65} + {}_{10}E_{45}\bar{A}_{55}\right) + 500\ {}_{20}E_{45}.\end{aligned}$$

   But under **UDD** $\bar{A}_x = \frac{i}{\delta}A_x$. Therefore

$$\begin{aligned}\mathbf{NSP} &= \frac{i}{\ln(1+i)}\left(1000A_{45} - 500\left({}_{20}E_{45}A_{65} + {}_{10}E_{45}A_{55}\right)\right) + 500\ {}_{20}E_{45} \\ &= \frac{0.06}{\ln(1.06)}\left(201.20 - 0.5\left(0.25634 \times 439.8 + 0.52652 \times 305.14\right)\right) + 0.5 \times 256.34 \\ &= \mathbf{194.59}.\end{aligned}$$

2. The expected present value or $\mathbf{APV(FB)}_0$ can be calculated as follows

$$\begin{aligned}\mathbf{APV(FB)}_0 &= 30vq_x + 35v^2p_xq_{x+1} + 40v^3\ {}_2p_x\ q_{x+2} + 40v^3\ {}_3p_x \\ &= 30vq_x + 35v^2p_xq_{x+1} + 40v^3p_xp_{x+1}\left(q_{x+2} + p_{x+2}\right) \\ &= 30vq_x + 35v^2\left(1 - q_x\right)q_{x+1} + 40v^3\left(1 - q_x\right)\left(1 - q_{x+1}\right) \\ &= \frac{30}{1.04}(0.02) + \frac{30}{(1.04)^2}(0.98)(0.04) + \frac{40}{(1.04)^3}(0.98)(0.96) \\ &= \mathbf{35.119}\end{aligned}$$