King Saud University
College of Sciences
Mathematics Department

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Solution of the final exam ACTU–362+372 Spring 2020 (20%)

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Problem 1. **(5 marks)**

1. Given $\mu_{40.5} = 1.35$ calculate $\mu_{40.25}$ and $\mu_{40.75}$ assuming UDD between integral ages.
2. You are given $\int_0^n {}_s p_{40}\, ds = 30.352$ and $\mu_{40+t} = \frac{0.5}{50-t}$ for all $t < 50$. Find n.
3. Assuming UDD between integral ages you are given: x is an integer and $0 < s < 1$ such that ${}_{0.25}p_{x+0.3} = 0.8$ and ${}_s p_{x+0.5} = 0.8$. Find s.
4. A life, age 65, is subject to mortality as described in the following excerpt from a 3–year select–and–ultimate table:

x	$\ell_{[x]}$	$\ell_{[x]+1}$	$\ell_{[x]+2}$	ℓ_{x+3}
65	5,000	4,750	4,500	4,200
66	4,800	4,550	4,250	3,800

Complete the following table

x	$q_{[x]}$	$q_{[x]+1}$	$q_{[x]+2}$
65			
66			

Solution:

1. Under UDD, we have $\mu_{x+r} = \frac{q_x}{1-rq_x}$ for all $0 < r < 1$, so $\mu_{40.5} = 1.35 = \frac{q_{40}}{1-0.5q_{40}}$ hence $q_{40} = 0.80597$, Thus

$$\mu_{40.25} = \frac{0.80597}{1-0.25\times 0.80597} = \mathbf{1.0093} \quad \text{and} \quad \mu_{40.75} = \frac{0.80597}{1-0.75\times 0.80597} = \mathbf{2.0377}$$

2. We have

$${}_s p_{40} = e^{-\int_0^s \frac{0.5}{50-u}du} = e^{0.5\ln\left(\frac{50-s}{50}\right)} = \sqrt{1-\frac{s}{50}},$$

thus

$$\int_0^n \sqrt{1-\frac{s}{50}}ds = \frac{100}{3}\left(1-\left(1-\frac{n}{50}\right)^{\frac{3}{2}}\right) = 30.352$$

which gives $n = \mathbf{40}$.

3. Under UDD, we know

$${}_{0.25}q_{x+0.3} = \frac{0.25q_x}{1-0.3q_x} = 0.2$$

which gives $q_x = 0.64516$ and

$${}_s p_{x+0.5} = \frac{1}{5} = \frac{sq_x}{1-0.5q_x} = s\frac{0.64516}{1-0.5\times 0.64516} = 0.2,$$

hence $s = \mathbf{0.21}$.

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4. We shall use $q_{[x]+k} = 1 - \frac{\ell_{[x]+k+1}}{\ell_{[x]+k}}$ for $k = 0,\ 1,\ 2.$

x	$q_{[x]}$	$q_{[x]+1}$	$q_{[x]+2}$
65	$1 - \dfrac{475}{500} = 1 - \dfrac{19}{20} = $ **0.05**	$1 - \dfrac{450}{475} = \dfrac{1}{19} = $ **0.05263**	$1 - \dfrac{42}{45} = \dfrac{1}{15} = $ **0.06667**
66	$1 - \dfrac{455}{480} = \dfrac{5}{96} = $ **0.05208**	$1 - \dfrac{425}{455} = \dfrac{6}{91} = $ **0.06593**	$1 - \dfrac{380}{425} = \dfrac{9}{85} = $ **0.10588**

<u>**Problem 2.**</u> **(5 marks)**

1. Calculate p_{70} given 1000 $A_{70} = 516$, 1000 $A_{71} = 530$ and $v = 0.95$
2. Calculate $10^5 A^{1\ (2)}_{40:\overline{2}|}$ using the following information: $i = 0.04$, $p_{40} = 0.8$ and $p_{41} = 0.75$ and assuming constant force of mortality between integral ages.
3. A life annuity of 1 on (30), is payable at the beginning of each year until age 60. The annuity payments are certain for the first 10 years. Calculate the actuarial present value of this annuity using ILT with $i = 6\%$..
4. An actuary uses Woolhouse's formula with three terms to approximates values of $\ddot{a}^{(2)}_{60} = 10.25$ and $\ddot{a}^{(4)}_{60} = 10.05$. Use the same formula, same mortality and interest rate assumptions as the actuary to calculate $\ddot{a}^{(12)}_{60}$.

Solution:

1. From whole life insurance recursion we have

$$A_{70} = v\, q_{70} + v\ p_{70}\ A_{71} = v\,(1 - \ p_{70}) + v\ p_{70}\ \ A_{71} = v - p_{70}\ v\,(1 - A_{71})$$

thus $p_{70} = \frac{v - A_{70}}{v(1 - A_{71})} = \frac{0.95 - 0.516}{0.95(1 - 0.530)} = \mathbf{0.972}$

2. The actuarial present value of future benefits is given by

$$A^{1\ (2)}_{45:\overline{2}|} = v^{\frac{0}{2}+\frac{1}{2}}\ {}_{\frac{0}{2}|\frac{1}{2}}q_{40} + v^{\frac{1}{2}+\frac{1}{2}}\ {}_{\frac{1}{2}|\frac{1}{2}}q_{40} + v^{\frac{2}{2}+\frac{1}{2}}\ {}_{\frac{2}{2}|\frac{1}{2}}q_{40} + v^{\frac{3}{2}+\frac{1}{2}}\ {}_{\frac{3}{2}|\frac{1}{2}}q_{40}$$

$$\begin{aligned} {}_{\frac{0}{2}|\frac{1}{2}}q_{40} &= {}_{0.5}q_{45} = 1 - \ {}_{0.5}p_{40} = 1 - p_{40}^{0.5} = 1 - 0.8^{0.5} = 0.10557, \\ {}_{0.5|0.5}q_{40} &= {}_{0.5}p_{40} - p_{40} = p_{40}^{0.5} - p_{40} = 0.8^{0.5} - 0.8 = 0.094427, \\ {}_{1|0.5}q_{40} &= p_{40}\,(1 - \ {}_{0.5}p_{41}) = p_{40}\left(1 - p_{41}^{0.5}\right) = 0.8\left(1 - 0.75^{0.5}\right) = 0.10718, \\ {}_{\frac{3}{2}|\frac{1}{2}}q_{40} &= {}_{1.5}p_{40} - \ {}_{2}p_{40} = p_{40}\ {}_{0.5}p_{41} - p_{40}p_{41} \\ &= p_{40}\left(p_{41}^{0.5} - p_{41}\right) = 0.8\left(0.75^{0.5} - 0.75\right) = 0.09282 \end{aligned}$$

hence

$$10^5 A^{1\ (2)}_{45:\overline{2}|} = 10^5\left(\frac{0.10557}{1.04^{0.5}} + \frac{0.094427}{1.04} + \frac{0.10718}{1.04^{1.5}} + \frac{0.09282}{1.04^2}\right) = \mathbf{38119}.$$

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Actuarial Mathematical Models I

3. This annuity is the sum of a 10–year annuity–certain and a 10–year deferred 20–year temporary life annuity on (30). So APV(of the Annuity) is

$$
\begin{aligned}
\ddot{a}_{\overline{10}|} + \ddot{a}_{[}10|]30{:}\overline{20}| &= \ddot{a}_{\overline{10}|} + {}_{10}E_{30}\ \ddot{a}_{40:\overline{20}|} \\
&= \ddot{a}_{\overline{10}|} + {}_{10}E_{30}\left(\ddot{a}_{40} - {}_{20}E_{40}\ \ddot{a}_{60}\right) \\
&= \frac{1-v^{10}}{d} + {}_{10}E_{30}\left(\ddot{a}_{40} - {}_{20}E_{40}\ \ddot{a}_{60}\right) \\
&= \frac{1-(0.9434)^{10}}{1-0.9434} + 0.54733\left(14.8166 - 0.27414 \times 11.1454\right) = \mathbf{14.239}.
\end{aligned}
$$

4. Remember that the Woolhouse's formula with three terms for a m–thly whole life annuity is

$$
\ddot{a}_x^{(m)} \simeq \ddot{a}_x - \frac{m-1}{2m} - \frac{m^2-1}{12m^2}\left(\mu_x + \delta\right).
$$

By assumption we have

$$
\begin{aligned}
\ddot{a}_{60}^{(2)} &= \ddot{a}_{60} - \frac{2-1}{4} - \frac{2^2-1}{12 \times 2^2}\left(\mu_{60}+\delta\right) = \ddot{a}_{60} - 0.250 - 0.0625\left(\mu_{60}+\delta\right) = 10.25 \\
\ddot{a}_{60}^{(4)} &= \ddot{a}_{60} - \frac{4-1}{8} - \frac{4^2-1}{12 \times 4^2}\left(\mu_{60}+\delta\right) = \ddot{a}_{60} - 0.375 - 0.0781\left(\mu_{60}+\delta\right) = 10.05
\end{aligned}
$$

which leads to $\ddot{a}_{60} = 10.8005$ and $\mu_{60} + \delta = 4.8077$, therefore

$$
\begin{aligned}
\ddot{a}_{60}^{(12)} &= \ddot{a}_{60} - \frac{12-1}{24} - \frac{12^2-1}{12^3}\left(\mu_{60}+\delta\right) \\
&= 10.8005 - \frac{12-1}{24} - \frac{12^2-1}{12^3}(4.8077) = \mathbf{9.9443}
\end{aligned}
$$

Problem 3. (5 marks)

1. A life purchases a special fully discrete 4–year term insurance policy. The benefit under this insurance, payable at the end of the year of death as given in the following table

Year of Death	Benefit Payment	Probability of Death
1	400	0.1
2	300	0.2
3	200	0.3
4	100	0.4

The effective interest rate is 4% per year and level premiums are payable annually at the beginning of each policy year. Calculate the level net annual premium of this insurance using E.P.

2. A life aged 30, purchases a 20–year level continuous **payment**, continuous whole life insurance policy with a benefit of 80,000. The insured is subject to a constant force of mortality equal to 0.075 and a constant force of interest equal to 0.025. Determine the net premium rate for this policy.

3. A special temporary 3–year life annuity–due on (40) pays 10,000, 15,000 and 20,000 at the beginning of year 1, 2, 3. Given: (i) ${}_tp_{40} = (0.7)^t$, for $0 \le t \le 4.5$ and ${}_tp_{40} = 0$, for $t > 4.5$ ii) $i = 0.05$, compute the actuarial present value of this annuity.

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Actuarial Mathematical Models I

4. For a life age (40) whose mortality follows $\mu_x = 0.008$ with $\delta = 0.04$. All the policies are fully continuous. **Find** the 20^{th}–percentile premium for:

   (a) a 30–year payment whole life insurance of 10^5;

   (b) a 20–year payment whole life insurance of 10^5

Solution:

1. The actuarial present value of the future benefits

$$\begin{aligned}\mathbf{APV(F.B.)}_0 &= 100\left(4v\ q_x + 3v^2\ {}_{1|}q_x + 2v^3\ {}_{2|}q_x + v^4\ {}_{3|}q_x\right)\\ &= 100\left(4v\ q_x + 3v^2\ p_x\ q_{x+1} + 2v^3\ {}_2p_x\ q_{x+2} + v^4\ {}_3p_x\ q_{x+3}\right)\\ &= 100\left(4v\ q_x + 3v^2\ p_x\ q_{x+1} + 2v^3\ p_x\ p_{x+1}\ q_{x+2} + v^4\ p_x\ p_{x+1}\ p_{x+2}\ q_{x+3}\right)\\ &= 100\left(4\frac{0.1}{1.04} + 3\frac{0.2}{(1.04)^2}0.9 + 2\frac{0.3}{(1.04)^3}0.9\times 0.8 + \frac{0.4}{(1.04)^4}0.9\times 0.8\times 0.7\right)\\ &= 144.03.\end{aligned}$$

and the actuarial present value of the future premiums

$$\begin{aligned}\mathbf{APV(F.P.)}_0 &= P\left(1 + v\ p_x + v^2\ {}_2p_x + v^3\ {}_3p_x\right) = P\left(1 + v\ p_x + v^2\ p_x\ p_{x+1} + v^3\ p_x\ p_{x+1}\ p_{x+2}\right)\\ &= P\left(1 + \frac{0.9}{1.04} + \frac{0.9\times 0.8}{(1.04)^2} + \frac{0.9\times 0.8\times 0.7}{(1.04)^3}\right) = 2.9791P\end{aligned}$$

By E.P. $P = \frac{144.03}{2.9791} = \mathbf{48.3468}$

2. The actuarial present value of the future benefits

$$\mathbf{APV(F.B.)}_0 = 80000\bar{A}_{30} = 80000\frac{0.075}{0.075+0.025} = 60,000.$$

and the actuarial present value of the future premiums

$$\begin{aligned}\mathbf{APV(F.P.)}_0 &= P\bar{a}_{30:\overline{20}|} = P\int_0^{20} e^{-0.025t}\ {}_tp_{30}dt\\ &= P\int_0^{20} e^{-0.025t}e^{-0.075t}dt = P\int_0^{20} e^{-0.1t}dt\\ &= P\frac{1-e^{-2}}{0.1} = 10P\left(1-e^{-2}\right) = 8.6466P.\end{aligned}$$

So the net premium rate $P = \frac{60000}{8.6466} = \mathbf{6939.14371}$.

3. Denote by α the expected present value or the actuarial present value of this annuity, which can be obtained using

$$\alpha = 1000\left(10q_{40} + (10+15v)\,p_{40}\ q_{41} + \left(10+15v+20v^2\right)\ {}_2p_{40}\ q_{42} + 20v^2\ {}_2p_{40}\right)$$

we know that

$${}_2p_{40} = p_{40}\ p_{41},\ \text{then } p_{41} = \frac{{}_2p_{40}}{p_{40}} = 0.7\ \text{ and }\ {}_3p_{40} = {}_2p_{40}\ p_{42},\ \text{then } p_{42} = \frac{{}_3p_{40}}{{}_2p_{40}} = 0.7$$

Therefore

$$\begin{aligned}\alpha &= 1000\left(10\times 0.3 + \left(10+\frac{15}{1.05}\right)(0.7)(0.3) + \left(10+\frac{15}{1.05}+\frac{20}{(1.05)^2}\right)(0.7)^2(0.3) + \frac{20}{(1.05)^2}(0.7)^2\right)\\ &= \mathbf{23226}.\end{aligned}$$

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4. The distribution of T_{40} is exponential with parameter 0.04. We know that the c.d.f. $F_{40}(t) = 1 - e^{-0.008t}$. Solving $F_{40}(t_{0.2}) = 0.2$, we get $t_{0.2} = 27.893$.

   (a) Then the 20^{th} –percentile premium for a 30–year payment whole life insurance of 10^5 on (40) is given

   $$P_{0.2} = \frac{S}{\bar{s}_{\overline{t_{0.2}}|}} = \frac{10^5}{\frac{e^{0.04\times 27.893}-1}{0.04}} = \mathbf{1949.54133}.$$

   since $t_{0.2} < 30$.

   (b) Then the 20^{th} –percentile premium for a 20–year payment whole life insurance of 10^5 on (40) is given

   $$P_{0.2} = \frac{S}{\bar{s}_{\overline{t_{0.2}}|}} = 10^5 \frac{e^{-0.04\times 27.893}}{1 - e^{-0.04\times 20}} 0.04 = \mathbf{2380.2179}.$$

Prof. Mohamed Eddahbi

Problem 4. **(5 marks)**

1. For a fully discrete whole life insurance of 2000 on (45), you are given $\mu_{45+t} = 0.02$ for $t < 10$ and $\mu_{45+t} = 0.04$ for $t \geq 10$ and the annual effective interest rate $i = 0.05$. The net level premium is 54.0402.

   (a) Calculate ${}_{10}V$ using prospective **and** retrospective methods

   (b) Use recursion de calculate ${}_{10.5}V$.

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2. For a 20–year endowment insurance of 4000 on (30), we assume that $\ell_x = 10\,(100 - x)$, for $0 \leq x \leq 100$, $i = 0.03$, benefits are payable **at the moment of death**, deaths are uniformly distributed between integral ages. and premiums are payable annually at **the beginning of each policy year**.
   Calculate the net premium reserve at time 10 for the insurance.

3. Consider a 10–year **term** insurance policy of 150,000 issued to a life aged 25. The force of mortality is $\mu_x = 0.005 + 0.0004e^{0.08x}$ and the force of interest is $\delta = 0.05$. The level premium rate is $P = 1402.8658$.

   (a) Write down the Thiele's differential equation satisfied by the net reserve ${}_tV$.

   (b) Use Euler's method

   $${}_{t+h}V \simeq {}_tV + h\left(P_t + \left(\delta_t + \mu_{x+t}\right)\ {}_tV - b_t\ \mu_{x+t}\right)$$

   with $h = 1$ and a backward recursion to find for the reserve at the end of year 7.

Actuarial Mathematical Models I

Solution:

1. Set $p_1 = e^{-0.02}$ and $p_1 = e^{-0.04}$, thus $q_1 = 1 - p_1 = 0.0198$ and $q_2 = 1 - p_2 = 0.03921$

   (a) By the **prospective method** we have

   $$\begin{aligned} {}_{10}V &= 2000A_{55} - P\ddot{a}_{55} = 2000\frac{q_2}{q_2 + i} - 54.0402\frac{1+i}{q_2 + i} \\ &= 2000\frac{1 - e^{-0.04}}{1.05 - e^{-0.04}} - 54.0402\frac{1.05}{1.05 - e^{-0.04}} = \mathbf{243.0084}. \end{aligned}$$

By the **retrospective method** we have

$$\begin{aligned}
{}_{10}V &= \frac{P\ddot{a}_{45:\overline{10}|} - 2000A^{1}_{45:\overline{10}|}}{{}_{10}E_{45}} = \frac{54.0402\frac{1+i}{q_1+i}\left(1-{}_{10}E_{45}\right) - 2000\frac{q_1}{q_1+i}\left(1-{}_{10}E_{45}\right)}{{}_{10}E_{45}} \\
&= \left(54.0402\,(1+i) - 2000q_1\right)\frac{\left(1-{}_{10}E_{45}\right)}{(q_1+i)\,{}_{10}E_{45}} \\
&= \left(54.0402\times 1.05 - 2000\left(1-e^{-0.02}\right)\right)\frac{\left(1-(1.05)^{-10}e^{-0.2}\right)}{\left(1-e^{-0.02}+0.05\right)(1.05)^{-10}e^{-0.2}} \\
&= \left(54.0402\times 1.05 - 2000\left(1-e^{-0.02}\right)\right)\frac{(1.05)^{10}e^{0.2}-1}{1.05-e^{-0.02}} = \mathbf{242.9784}.
\end{aligned}$$

(b) Now, by **recursion** we can write

$$\left({}_{10}V + P\right)(1+i)^{0.5} = v^{0.5}2000\;{}_{0.5}q_{55} + {}_{10.5}V\;{}_{0.5}p_{55}$$

hence

$$\begin{aligned}
{}_{10.5}V &= \frac{\left({}_{10}V + P\right)(1+i)^{0.5} - v^{0.5}2000\left(1-{}_{0.5}p_{55}\right)}{{}_{0.5}p_{55}} \\
&= \frac{(264.13+52.245)\,(1.05)^{0.5} - (1.05)^{-0.5}\,2000\left(1-e^{-0.04\times 0.5}\right)}{e^{-0.04\times 0.5}} = \mathbf{291.31}.
\end{aligned}$$

2. By the prospective method we find

$${}_{10}V = 4000\left(\bar{A}_{40:\overline{10}|} - \frac{\bar{A}_{30:\overline{20}|}}{\ddot{a}_{30:\overline{20}|}}\;\ddot{a}_{40:\overline{10}|}\right)$$

Moreover

$$\begin{aligned}
\bar{A}_{40:\overline{10}|} &= \bar{A}^{1}_{35:\overline{10}|} + {}_{10}E_{40} = \frac{\bar{a}_{\overline{10}|}}{100-40} + \frac{1}{1.03^{10}}\frac{100-40-10}{100-40} \\
&= \frac{1}{\ln(1.03)}\frac{1-1.03^{-10}}{60} + \frac{1}{1.03^{10}}\frac{50}{60} = 0.76437
\end{aligned}$$

and

$$\begin{aligned}
\bar{A}_{30:\overline{20}|} &= \bar{A}^{1}_{30:\overline{20}|} + {}_{20}E_{30} = \frac{\bar{a}_{\overline{20}|}}{100-30} + \frac{1}{1.03^{20}}\frac{100-30-20}{100-30} \\
&= \frac{1}{\ln(1.03)}\frac{1-1.03^{-20}}{70} + \frac{1}{1.03^{20}}\frac{50}{70} = 0.61119.
\end{aligned}$$

For annuities we have

$$\begin{aligned}
\ddot{a}_{40:\overline{10}|} &= \frac{1+i}{q+i}\left(1-{}_{10}E_{40}\right) = \frac{1.03}{\frac{1}{60}+0.03}\left(1-\frac{1}{1.03^{10}}\frac{100-40-10}{100-40}\right) = 8.3854 \\
\ddot{a}_{30:\overline{20}|} &= \frac{1+i}{q+i}\left(1-{}_{20}E_{30}\right) = \frac{1.03}{\frac{1}{70}+0.03}\left(1-\frac{1}{1.03^{20}}\frac{100-30-20}{100-30}\right) = 14.060.
\end{aligned}$$

Therefore ${}_{10}V = 4000\left(0.76437 - \frac{0.61119}{14.060}8.3854\right) = \mathbf{1599.42235}$.

3. We have $P = 1402.8658$.

Prof. Mhamed Eddahbi meddahbi@ksu.edu.sa Actuarial Mathematical Models I

(a) The Thiele's differential equation satisfied by the benefit reserve is

$$\begin{aligned}\frac{d\ {}_tV}{dt} &= P_t + (\delta + \mu_{25+t})\ {}_tV - b_t\ \mu_{25+t} \\ &= 1402.8658 + (0.05 + 0.005 + 0.0004e^{0.08(25+t)})\ {}_tV - 15\left(50 + 4e^{0.08(25+t)}\right) \\ &= 1402.8658 - 15\left(50 + 4e^{0.08(25+t)}\right) + (0.05 + 0.005 + 0.0004e^{0.08(25+t)})\ {}_tV \\ &= 652.8658 - 60e^{0.08(25+t)} + (0.055 + 0.0004e^{0.08(25+t)})\ {}_tV\end{aligned}$$

(b) The backward recursion scheme of Euler's method is

$${}_tV \simeq \frac{{}_{t+h}V - h\left(P_t - b_t\ \mu_{x+t}\right)}{1 + h\left(\delta_t + \mu_{x+t}\right)} = \frac{{}_{t+h}V - h\left(652.8658 - 60e^{0.08(25+t)}\right)}{1 + h(0.055 + 0.0004e^{0.08(25+t)})}$$

with terminal value ${}_{10}V = 0$. Now we start with $t = 9$ and compute the reserves recursively, using $h = 1$, we get

$$\begin{aligned}{}_9V &= \frac{{}_{10}V - \left(652.8658 - 60e^{0.08(34)}\right)}{1 + (0.055 + 0.0004e^{0.08(34)})} = \frac{0 - \left(652.8658 - 60e^{0.08(34)}\right)}{1 + (0.055 + 0.0004e^{0.08(34)})} = \mathbf{243.10650}, \\ {}_8V &= \frac{{}_9V - \left(652.8658 - 60e^{0.08(33)}\right)}{1 + (0.055 + 0.0004e^{0.08(33)})} = \frac{243.10650 - \left(652.8658 - 60e^{0.08(33)}\right)}{1 + (0.055 + 0.0004e^{0.08(33)})} = \mathbf{406.40277}, \\ {}_7V &= \frac{{}_8V - \left(652.8658 - 60e^{0.08(32)}\right)}{1 + (0.055 + 0.0004e^{0.08(32)})} = \frac{406.40277 - \left(652.8658 - 60e^{0.08(32)}\right)}{1 + (0.055 + 0.0004e^{0.08(32)})} = \mathbf{499.62161}.\end{aligned}$$

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Actuarial Mathematical Models I