Chapter 4

Laplace Transform

Introduction

Pierre-Simon Laplace: French Scholar (1749 –1827)

Fourier transforms involve purely imaginary complex exponentials: $e^{st}, \; s = j\omega$

Laplace transforms involve complex exponentials: $e^{st}, \; s = \sigma + j\omega$ $\quad x(t) \overset{L}{\leftrightarrow} X(s)$

Eigen-function property applies to any complex number S

Laplace Transform: $$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

$$x(t) \overset{\mathcal{L}}{\rightarrow} X(s)$$

$$X(s)\Big|_{s=j\omega} = \mathrm{F}\{x(t)\}$$ → Fourier Transform

Laplace Transform is a *generalization* of the continuous-time Fourier transform,

Laplace Transform and Fourier Transform

$$X(\sigma + j\omega) = \int_{-\infty}^{\infty} x(t)e^{-(\sigma+j\omega)t}dt = \int_{-\infty}^{\infty} x(t)e^{-\sigma t}e^{-j\omega t}dt = \int_{-\infty}^{\infty} x'(t)e^{-j\omega t}dt$$

The Laplace transform is the Fourier transform of the transformed signal $x'(t) = x(t)e^{-\sigma t}$

Example 1

Consider the signal $x(t) = e^{-at}u(t)$

The Fourier transform: $X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = \int_{0}^{\infty} e^{-at}e^{-j\omega t}dt = \dfrac{1}{a + j\omega}$ $, a > 0$

The Laplace transform: $X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt = \int_{0}^{\infty} e^{-at}e^{-(\sigma+j\omega)\,t}dt = \int_{0}^{\infty} e^{-(a+\sigma)t}e^{-j\omega t}dt$

which is the Fourier Transform of $e^{-(a+\sigma)t}u(t)$

$$X(\sigma + j\omega) = \frac{1}{(\sigma + a) + j\omega} \quad , \sigma + a > 0$$

Or

$$e^{-at}u(t) \xrightarrow{\mathcal{L}} \frac{1}{s + a} \quad , R_e\{s\} > -a$$

If *a* is negative or zero, the Laplace Transform still exists

Example 2

Consider the signal $x(t) = -e^{-at}u(-t)$

The Laplace transform is: $X(s) = -\int_{-\infty}^{\infty} e^{-at}e^{-st}u(-t)dt \quad = -\int_{-\infty}^{0} e^{-(s+a)t}dt \quad = \frac{1}{s+a}$

$$-e^{-at}u(-t) \xrightarrow{\mathcal{L}} \frac{1}{s+a} \quad , R_e\{s\} < -a$$

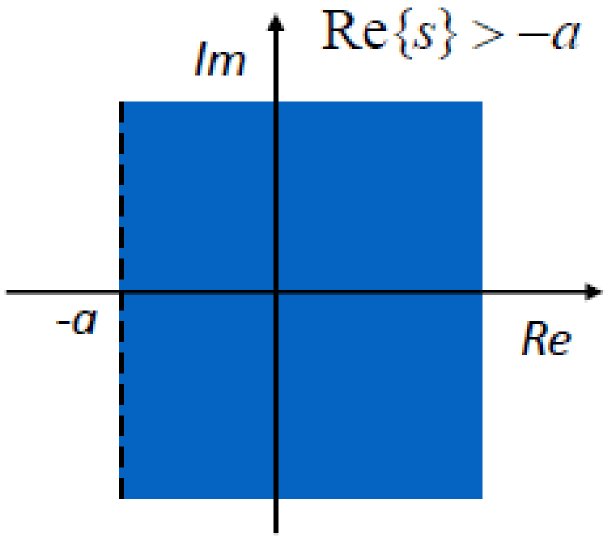
Convergence $Re\{s+a\} < 0 \; for \; t < 0$

- Convergence requires that $Re\{s+a\} < 0$ or $Re\{s\} < -a$
- In example 1: $$e^{-at}u(t) \xrightarrow{\mathcal{L}} \frac{1}{s+a} \quad , R_e\{s\} > -a$$
- The Laplace transform is identical for two different signals. However the regions of convergence of s are mutually exclusive (non-intersecting).
- For a Laplace transform, we need both the expression and the *Region Of Convergence* (ROC)

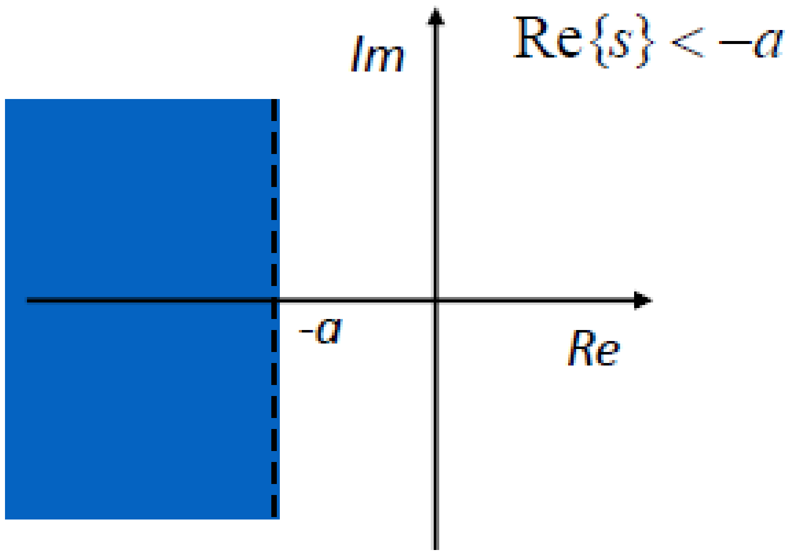
Region of Convergence For Laplace Transform

- The Fourier transform exists for most signals with *finite energy* (Dirichlet convergence conditions)
- The Region Of Convergence (ROC) of the Laplace Transform is the region of values for $s = \sigma + j\omega$ for which the Fourier transform of $x(t)e^{-\sigma t}$ converges.).

Example 1



Example 2



- The ROC of X(s) consists of strips (bands) parallel to the jω-axis in the s-plane. The shaded regions denote the ROC for the Laplace transform
- A complete specification of the Laplace transform requires the algebraic expression for X(s) and the associated ROC

Example 3

- Consider a signal that is the sum of two real exponentials:

$$x(t) = 3e^{-2t}u(t) - 2e^{-t}u(t)$$

- The Laplace transform is then: $X(s) = \int_{-\infty}^{\infty} \left[3e^{-2t}u(t) - 2e^{-t}u(t)\right]e^{-st}\,dt$

$$= 3\int_{-\infty}^{\infty} e^{-2t}u(t)e^{-st}\,dt - 2\int_{-\infty}^{\infty} e^{-t}u(t)e^{-st}\,dt$$

- Using Example 1, each expression can be evaluated as:

$$X(s) = \frac{3}{s+2} - \frac{2}{s+1}$$

- The ROC associated with these terms are Re{s}>-1 and Re{s}>-2. Therefore, both will converge for Re{s}>-1, and the Laplace transform:

$$X(s) = \frac{s-1}{s^2+3s+2}$$

Poles and Zeros

The Laplace transform $$X(s) = \frac{N(s)}{D(s)}$$

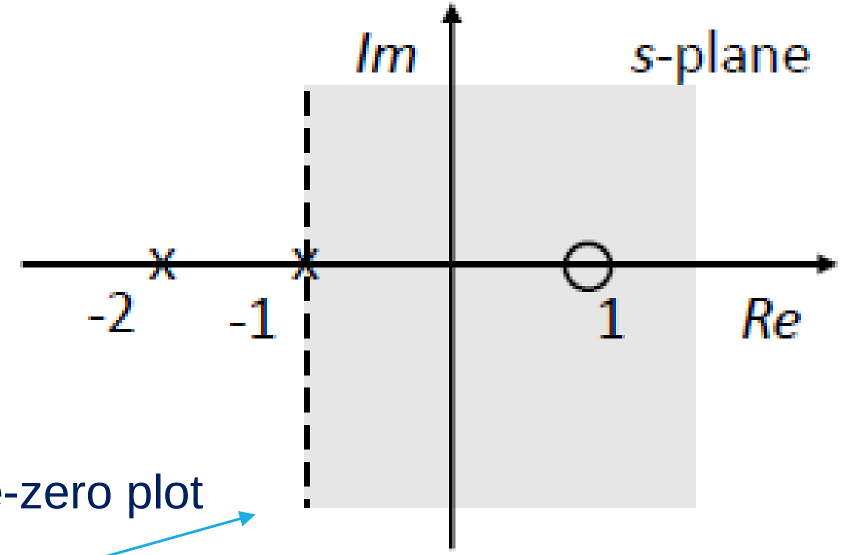
Zeros: roots of N(s) → Makes X(s) zero

Poles: roots of D(s) → Makes X(s) infinite

For the Laplace transform: $$X(s) = \frac{s-1}{(s+2)(s+1)}$$

Pole-zero plot

Pole-zero plot



O – Zero

x – Pole

Zero calculation:	Pole calculation:
$s - 1 = 0 \rightarrow s = 1$	$(s+2)(s+1) = 0 \rightarrow s = -2, -1$

Laplace transform X(s) is rational if it is a ratio of polynomials in the complex variable *s*.

$$X(s) = \frac{s}{s^2 - 2s + 1}$$ Rational

$$X(s) = \frac{e^t}{s^2 - 2s + 1}$$ Not Rational

Poles and Zeros at Infinity

- If the denominator polynomial order is greater than the numerator polynomial order, there are zeros at infinity. (their number is the difference in order).
- If the numerator polynomial order is greater than the denominator polynomial order, there are poles at infinity. (their number is the difference in order).

$$X(s) = \frac{(s-1)^2}{(s+1)(s-2)}$$

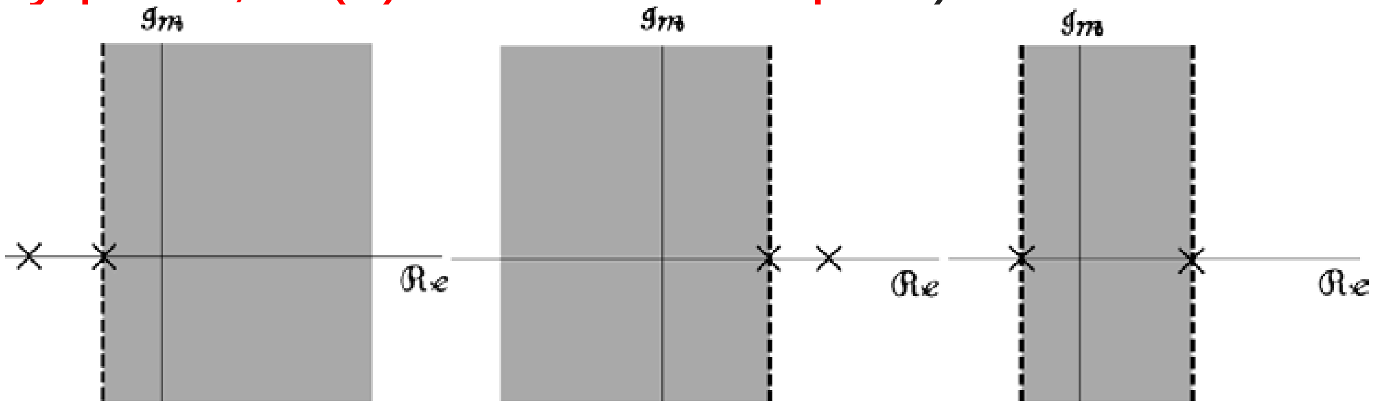
Neither poles or zeros are at infinity

$$X(s) = \frac{s-1}{s^2+3s+2}$$

One zero at infinity

- For rational Laplace transforms, the ROC is bounded by the poles (rightmost pole or leftmost pole but does not contain any poles, X(s) is infinite at a pole).

- If the ROC includes the $j\omega - axis$ then the Fourier transform exists. The Fourier transform is the evaluation of the Laplace transform along the $j\omega - axis$.



Right sided signal, rightmost pole

Left-sided signal, leftmost pole

Double-sided signal, Intersection of the two regions

Example 4

$$x(t) = e^{-2t}u(t) + e^{-t}(\cos 3t)u(t)$$

$$x(t) = \left[e^{-2t} + e^{-t}\frac{e^{3jt}+e^{-3jt}}{2}\right]u(t) = \left[e^{-2t} + \frac{e^{-(1-3j)t}+e^{-(1+3j)t}}{2}\right]u(t)$$

From Exercise 1

$$x(t) = e^{-at}u(t) \overset{\mathcal{L}}{\Longleftrightarrow} \frac{1}{s+a}$$

$$t > 0 \rightarrow \Re e\{s\} > -a$$

$$X(s) = \frac{1}{s+2} + \frac{1}{2}\frac{1}{s+(1-3j)} + \frac{1}{2}\frac{1}{s+(1+3j)}$$

$Re\{s\} > -2$ $\quad$ $Re\{s\} > -1$ $\quad$ $Re\{s\} > -1$

After combining $\rightarrow$ $Re\{s\} > -1$

$$X(s) = \frac{1}{s+2} + \frac{1}{2}\frac{1}{s+(1-3j)} + \frac{1}{2}\frac{1}{s+(1+3j)}$$

Example 5

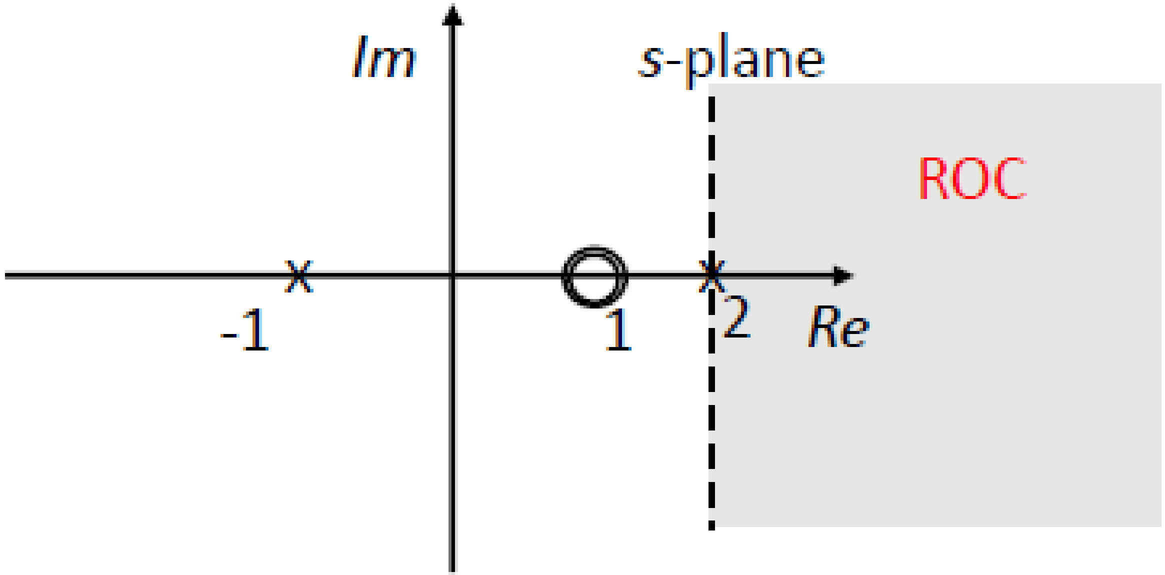
$$x(t) = \delta(t) - \frac{4}{3}e^{-t}u(t) + \frac{1}{3}e^{2t}u(t)$$

If x(t) is of finite duration and is absolutely integrable, ➡ ROC is the entire s-plane.

$\mathcal{L}\{\delta(t)\} = 1,$ ROC = Entire s-plane

$$X(s) = 1 - \frac{4}{3}\frac{1}{s+1} + \frac{1}{3}\frac{1}{s-2} = \frac{(s-1)^2}{(s+1)(s-2)}$$

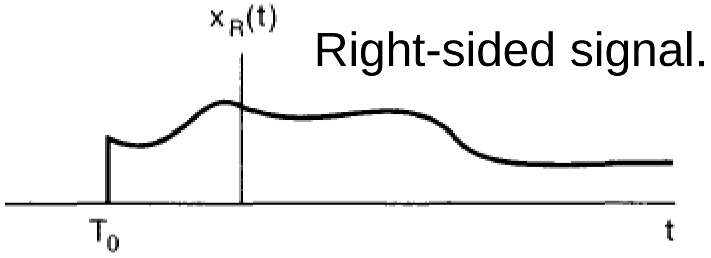
ROC: $Re\{s\} > 2$

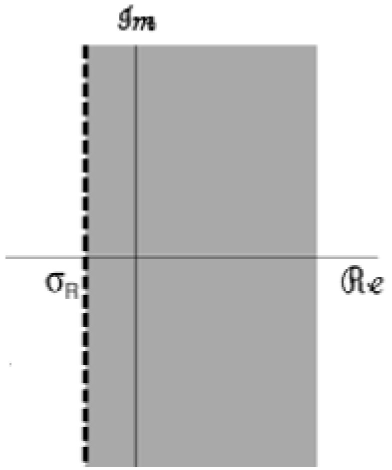


Two-sided Signals

If x(t) is right-sided, and if the line $\Re e\{s\} = \sigma_0$ is in the ROC

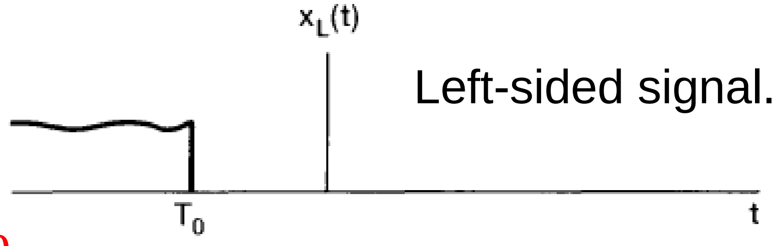
➡ The area of s for which $\Re e\{s\} > \sigma_0$ will also be in the ROC.

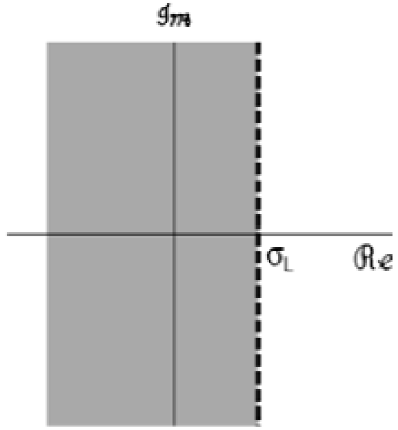




If x(t) is left-sided, and if the line $\Re e\{s\} = \sigma_0$ is in the ROC

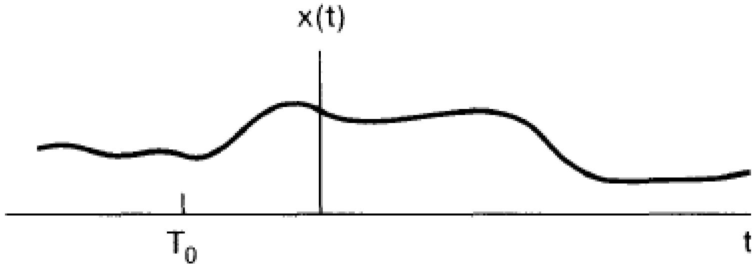
➡ The area of s for which $\Re e\{s\} < \sigma_0$ will also be in the ROC.

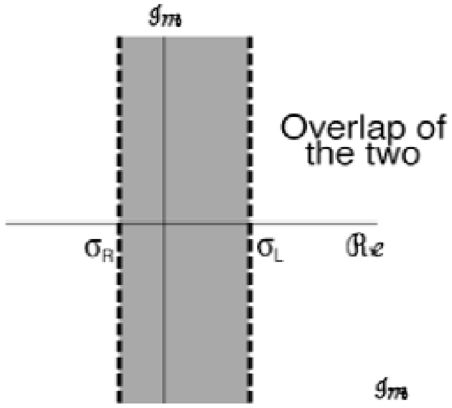




If x(t) is two-sided, and if the line $\Re e\{s\} = \sigma_0$ is in the ROC

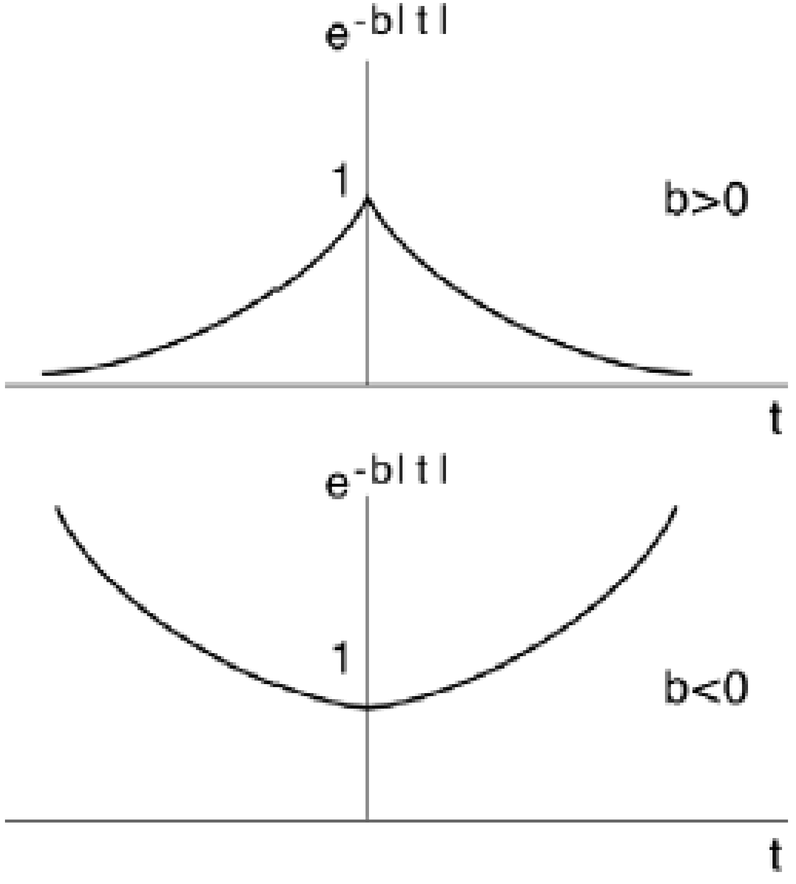
➡ The ROC is a trip in s-plane that includes $\Re e\{s\} = \sigma_0$





Example 6

$$x(t) = e^{-b|t|} = e^{bt}u(-t) + e^{-bt}u(t)$$ ← divided it into the sum of a right-sided and left-sided signal;

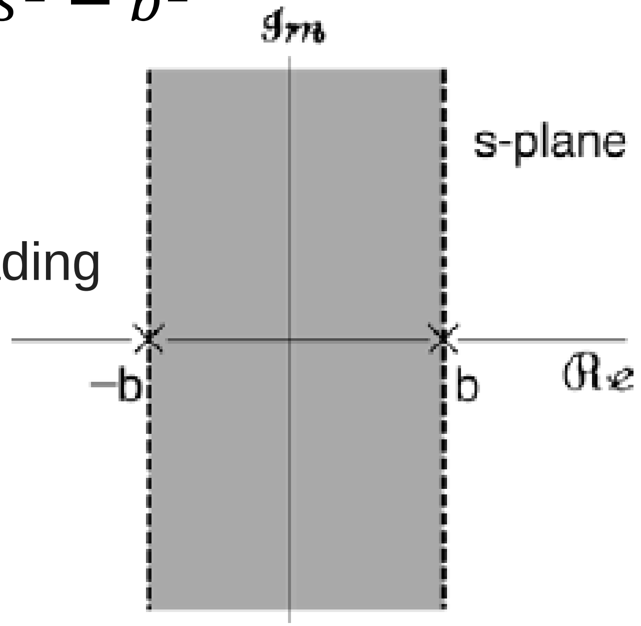


$$e^{-bt}u(t) \overset{\mathcal{L}}{\Leftrightarrow} \frac{1}{s+b} \quad \Re e\{s\} > -b$$

$$e^{+bt}u(-t) \overset{\mathcal{L}}{\Leftrightarrow} \frac{-1}{s-b} \quad \Re e\{s\} < +b$$

$$e^{-b|t|}u(-t) \overset{\mathcal{L}}{\Leftrightarrow} \frac{1}{s+b} - \frac{1}{s-b} = \frac{-2b}{s^2-b^2} \quad -b < \Re e\{s\} < +b$$

pole-zero plot with the shading indicating the ROC.



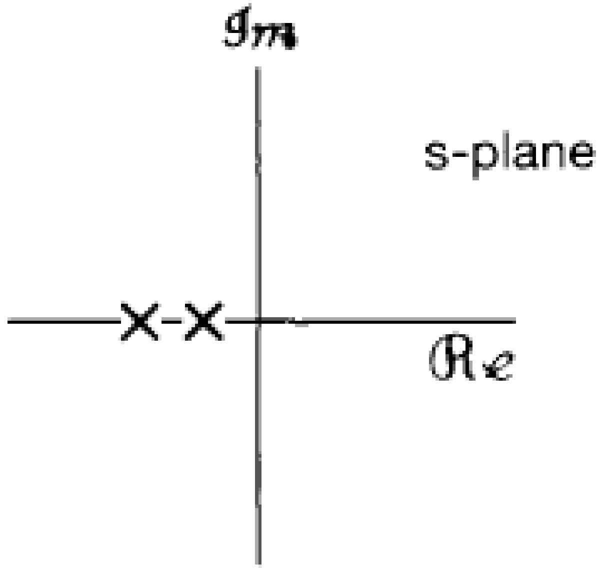
- If b < 0, the Laplace transform does not exist.
- Hence, the ROC plays an integral role in the Laplace transform.

Example 7

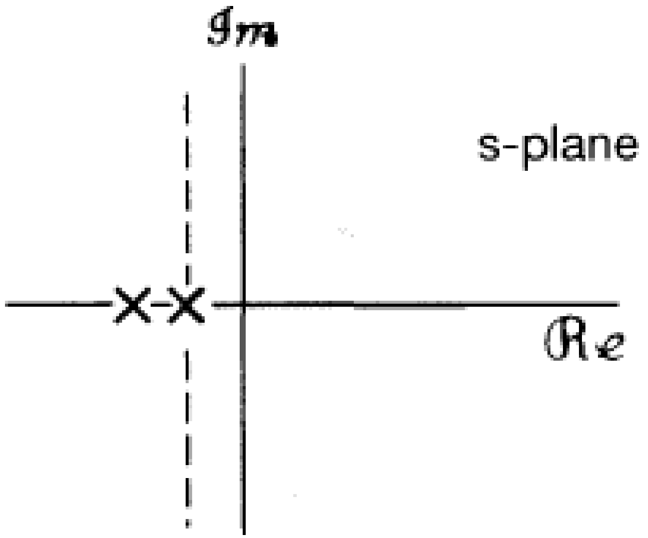
Let

$$X(s) = \frac{1}{(s+1)(s+1)}$$

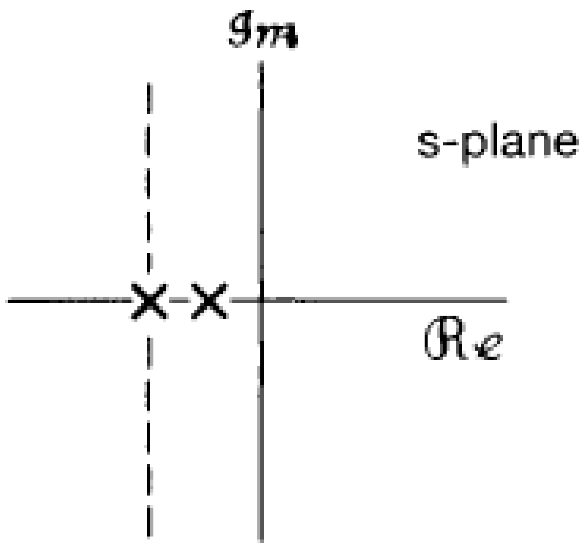
Pole-zero pattern



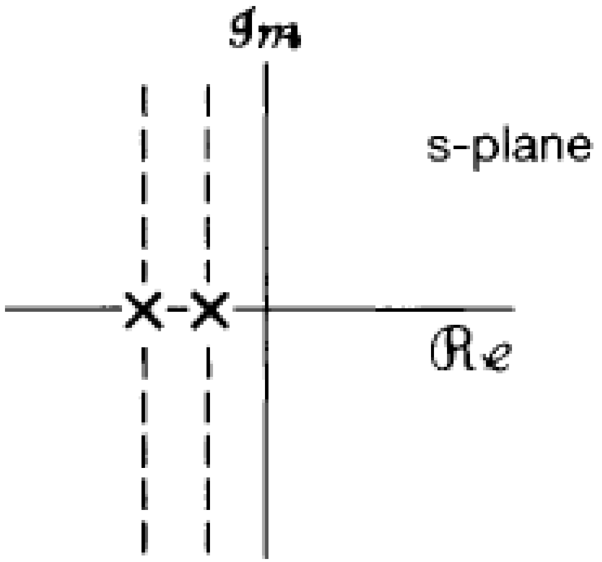
There are three possible ROCs associated with this expression corresponding to three distinct signals.



ROC corresponding to
a right-sided sequence



ROC corresponding to
a left-sided sequence



ROC corresponding to a
two-sided sequence

Inverse Laplace transform

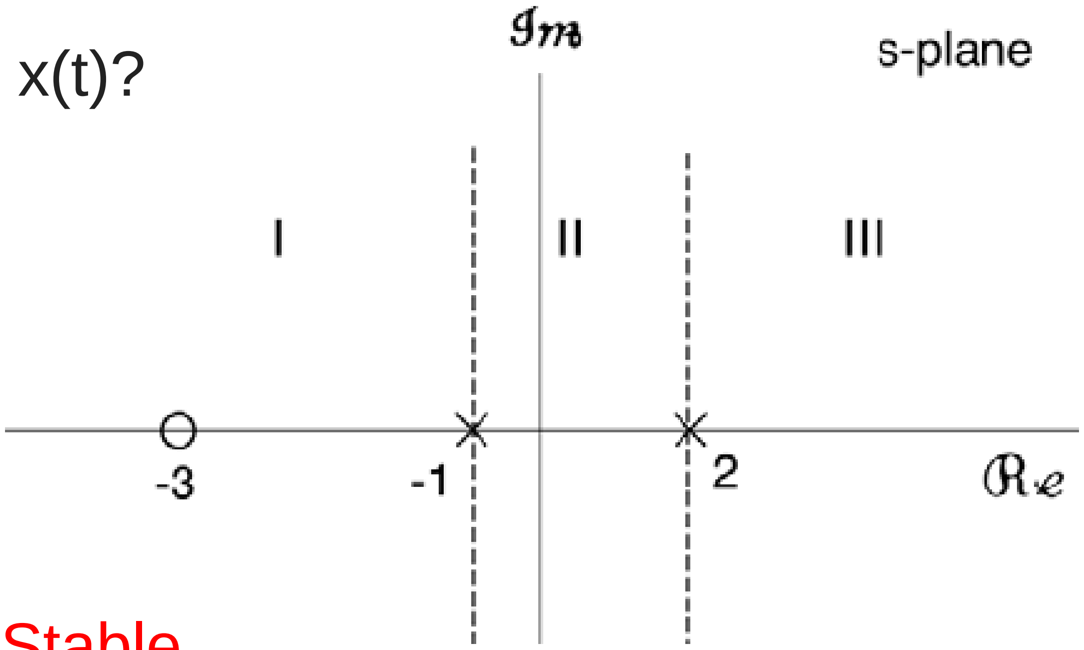
- we can recover x(t) from its Laplace transform evaluated for a set of values of $s = \sigma - j\omega$ in the ROC by varying ω from $-\infty$ to $+\infty$

$$x(t) = \frac{1}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} X(s)e^{st}ds$$

- The inverse Laplace transform can be determined using the technique of partial fraction expansion (easier method)

Example 8

- Consider the Laplace transform: $X(s) = \frac{(s+3)}{(s+1)(s-2)}$
- Can we uniquely determine the original signal, x(t)?



- There are three possible ROCs:
- ROC III: only if x(t)is right-sided. Causal, NOT Stable
- ROC I: only if x(t)is left-sided. NOT causal, NOT stable
- ROC II: only if x(t) has a Fourier transform. STABLE

Find x(t) for different ROCs

To obtain the inverse Laplace transform, we first perform a partial-fraction expansion

$$X(s) = \frac{(s+3)}{(s+1)(s-2)}$$

$$\frac{s+3}{(s+1)(s-2)} = \frac{A}{s+1} + \frac{B}{s-2} \Rightarrow s+3 = A(s-2) + B(s+1)$$

$$\Rightarrow s+3 = (A+B)s - 2A + B$$

$$\Rightarrow A+B = 1; -2A + B = 3$$

$$\Rightarrow A = -2/3, B = 5/3$$

$$X(s) = \frac{(s+3)}{(s+1)(s-2)} = -\frac{2}{3}\frac{1}{s+1} + \frac{5}{3}\frac{1}{s-2}$$

ROC III → Causal, not stable, x(t) right sided $x(t) = -\frac{2}{3}e^{-t}u(t) + \frac{5}{3}e^{2t}u(t)$

ROC I → Not causal, not stable, x(t) left sided $x(t) = +\frac{2}{3}e^{-t}u(-t) - \frac{5}{3}e^{2t}u(-t)$

ROC II → Not causal, stable, x(t) two sided $x(t) = -\frac{2}{3}e^{-t}u(t) - \frac{5}{3}e^{2t}u(-t)$

Example 9

Find inverse Laplace transform of $X(s)=\dfrac{1}{(s+1)(s+2)}$; Re{s}>-1

$$X(s)=\frac{1}{(s+1)(s+2)}=\frac{A}{s+1}+\frac{B}{s+2} \quad \Rightarrow A=1, B=-1 \Rightarrow X(s)=\frac{1}{s+1}-\frac{1}{s+2}$$

ILT

$$x(t)=\left[e^{-t}-e^{-2t}\right]u(t)$$

Find inverse Laplace transform of $X(s)=\dfrac{1}{(s+1)(s+2)}$; Re{s}<-2

$$X(s)=\frac{1}{(s+1)(s+2)}=\frac{A}{s+1}+\frac{B}{s+2} \quad \Rightarrow A=1, B=-1 \Rightarrow X(s)=\frac{1}{s+1}-\frac{1}{s+2}$$

ILT

$$x(t)=\left[-e^{-t}+e^{-2t}\right]u(-t)$$

PROPERTIES OF THE LAPLACE TRANSFORM

1. Linearity

IF $x_1(t) \overset{\mathcal{L}}{\Longleftrightarrow} X_1(s)$ $ROC = R_1$ AND $x_2(t) \overset{\mathcal{L}}{\Longleftrightarrow} X_2(s)$ $ROC = R_2$

THEN $a\, x_1(t) + b\, x_2(t) \overset{\mathcal{L}}{\Longleftrightarrow} a\, X_1(s) + b\, X_2(s)$ $ROC = R_1 \cap R_2$

---

2. Time Shifting

IF $x(t) \overset{\mathcal{L}}{\Longleftrightarrow} X(s)$ $ROC = R$

THEN $x(t - t_0) \overset{\mathcal{L}}{\Longleftrightarrow} e^{-st_0} X(s)$ $ROC = R$

Example 10 Linearity and Time Shifting

Consider the following signal, which is a linear sum of two time-shifted sinusoids.

$$x(t) = 2x_1(t-2.5) - 0.5x_1(t-4)$$
$$x_1(t) = \sin(\omega_0 t)u(t)$$

Laplace transform of $x_1(t)$: $X_1(s) = \dfrac{\omega_0}{s^2 + \omega_0{}^2}$; $\mathrm{Re}\{s\} > 0$

Using linearity and time-shifting properties of Laplace transform, we get:

$$X(s) \overset{\mathcal{L}}{\Longleftrightarrow} 2e^{-2.5\,s}\,X_1(s) - 0.5e^{-4\,s}\,X_1(s) \quad R_e\{s\} > 0$$

$$\Rightarrow \quad X(s) = \left(2e^{-2.5s} - 0.5e^{-4s}\right)\frac{\omega_0}{s^2 + \omega_0^2} \qquad \mathrm{Re}\{s\} > 0$$

3. Shifting in the s-Domain

IF $x(t) \overset{\mathcal{L}}{\Longleftrightarrow} X(s)$ $with\ ROC = R$ THEN $e^{s_0 t}x(t) \overset{\mathcal{L}}{\Longleftrightarrow} X(s - s_0)$

$with\ ROC = R + R_e\{s_0\}$

4. Time Scaling

IF $x(t) \overset{\mathcal{L}}{\Longleftrightarrow} X(s)$ $with\ ROC = R$ THEN $x(at) \overset{\mathcal{L}}{\Longleftrightarrow} \frac{1}{|a|}X\left(\frac{s}{a}\right)$ $with\ ROC = aR$

5. Conjugation

IF $x(t) \overset{\mathcal{L}}{\Longleftrightarrow} X(s)$ $with\ ROC = R$ THEN $x^*(t) \overset{\mathcal{L}}{\Longleftrightarrow} X^*(s^*)$ $with\ ROC = R$

6. Differentiation in the s-Domain

IF $x(t) \overset{\mathcal{L}}{\Longleftrightarrow} X(s)$ $with\ ROC = R$ THEN $-t\,x(t) \overset{\mathcal{L}}{\Longleftrightarrow} \frac{dX(s)}{ds}$ $with\ ROC = R$

7. Convolution

- The Laplace transform also has the multiplication property, i.e.

IF $x(t) \overset{\mathcal{L}}{\Longleftrightarrow} X(s)$ $\quad ROC = R_1$

AND $h(t) \overset{\mathcal{L}}{\Longleftrightarrow} H(s)$ $\quad ROC = R_2$

THEN $x(t) * h(t) \overset{\mathcal{L}}{\Longleftrightarrow} X(s)\,H(s)$ $\quad ROC = R_1 \cap R_2$

Convolution in time-domain becomes multiplication in Laplace domain.

- Note that pole-zero cancellation may occur between H(s) and X(s) which extends the ROC

$$X(s) = \frac{s+1}{s+2} \qquad \Re\{s\} > -2$$

$$H(s) = \frac{s+2}{s+1} \qquad \Re\{s\} > -1$$

$$\Rightarrow \quad X(s)H(s) = 1 \qquad -\infty < \Re\{s\} < \infty$$

8. Differentiation in Time-Domain

$$x(t) \overset{\mathcal{L}}{\Longleftrightarrow} X(s) \qquad ROC = R$$

The Inverse Laplace transform $$x(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}\,ds$$

The Derivative $$\frac{dx(t)}{dt} = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} sX(s)e^{st}\,ds$$

$$\Rightarrow \quad \frac{dx(t)}{dt} \overset{\mathcal{L}}{\Longleftrightarrow} sX(s)$$ ROC containing R

Problem 1

Determine the Laplace transform and ROC and pole-zero plot:

$$x(t) = e^{-2t}u(t) + e^{-3t}u(t)$$

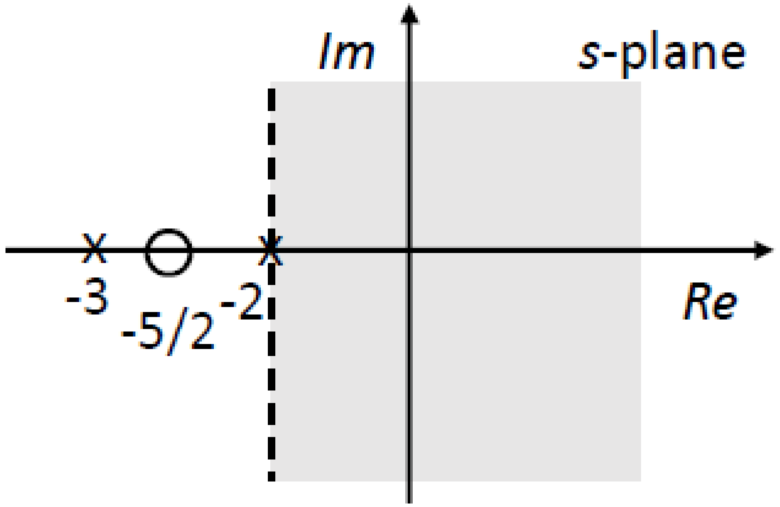
$Zero: -5/2$

$$X(s) = \frac{1}{s+2} + \frac{1}{s+3} = \frac{2s+5}{(s+2)(s+3)}$$

$Poles: -2, -3$

ROC: $R_e\{s\} > -2$

Because of $u(t)$, ROC is on the right-side of s-plane



CAUSAL: because X(s) is rational and ROC is on the right-side of the right-most pole

STABLE: because ROC contains imaginary axis ($j\omega$)

Problem 2

Determine the Laplace transform and ROC and pole-zero plot:

$$x(t) = e^{-4t}u(t) + e^{-5t}(\sin 5t)u(t)$$

$$x_1(t) = \sin(\omega_0 t)u(t) \longrightarrow X_1(s) = \frac{\omega_0}{s^2 + \omega_0^2}; \quad \mathrm{Re}\{s\} > 0$$

$$e^{s_0 t}x(t) \overset{\mathcal{L}}{\Longleftrightarrow} X(s - s_0)$$ Shifting in the s-domain

$$X(s) = \frac{1}{s+4} + \frac{5}{(s+5)^2 + 5^2} = \frac{1}{(s+4)} + \frac{5}{s^2 + 10s + 50} = \frac{s^2 + 15s + 70}{(s+4)(s^2 + 10s + 50)}$$

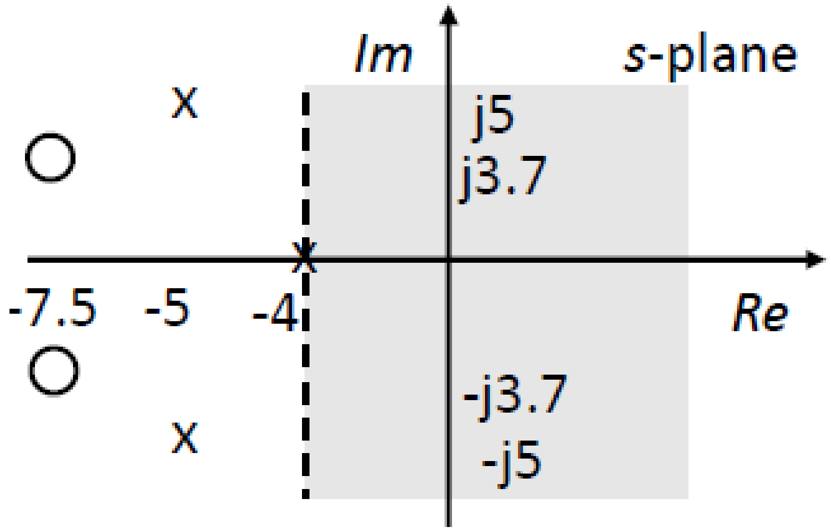
Poles: solving $s + 4 = 0 \ \rightarrow s = -4;$

$and \ \ s^2 + 10s + 50 = 0 \ \ \rightarrow s = -5 \pm j\,5$

Zeros: solving $s^2 + 15s + 70 = 0$

$\rightarrow s = -7.5 \pm j\,3.7$

$ROC\colon R_e\{s\} > -4$



CAUSAL
STABLE

Problem 3

Determine the Laplace transform and ROC and pole-zero plot:

$$x(t) = e^{2t}u(-t) + e^{3t}u(-t)$$

$$e^{+bt}u(-t) \overset{\mathcal{L}}{\Longleftrightarrow} \frac{-1}{s-b} \quad \Re e\{s\} < +b$$

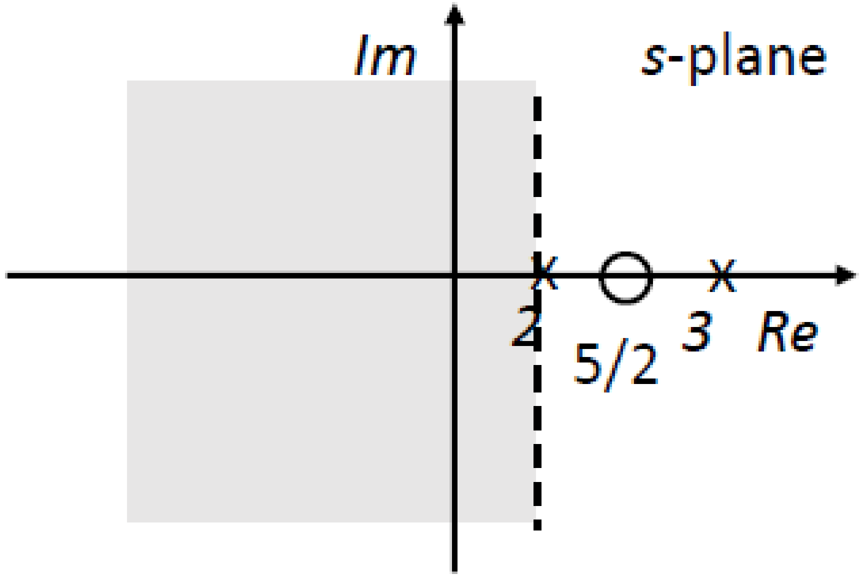
$$X(s) = -\frac{1}{s-2} - \frac{1}{s-3} = \frac{5-2s}{(s-2)(s-3)}$$

Zero: $5/2$

Poles: $2, 3$

$ROC: R_e\{s\} < 2$

Because of $u(-t)$, ROC is on the left-side of s-plane



NOT CAUSAL;
STABLE

Problem 4

Find inverse Laplace transform of $X(s) = \frac{1}{s^2+9}$

Laplace transform of $sin(\omega_0 t)u(t) \overset{\mathcal{L}}{\Leftrightarrow} \frac{\omega_0}{s^2+\omega_0{}^2}$ $ROC: R_e\{s\} > 0$

We have $X(s) = \frac{1}{s^2+9} = \frac{1}{s^2+3^2} = \frac{1}{3}\frac{3}{s^2+3^2}$

For ROC: Re{s} > 0,	$x(t) = \frac{1}{3}\sin(3t)u(t)$
For ROC: Re{s} < 0,	$x(t) = -\frac{1}{3}\sin(3t)u(-t)$

Problem 4

Find inverse Laplace transform of $X(s) = \dfrac{s}{s^2 + 9}$

Laplace transform of $cos(\omega_0 t)u(t) \overset{\mathcal{L}}{\Longleftrightarrow} \dfrac{s}{s^2 + {\omega_0}^2}$ $ROC: R_e\{s\} > 0$

We have $X(s) = \dfrac{s}{s^2 + 9} = \dfrac{s}{s^2 + 3^2}$

For ROC: Re{s} > 0,	$x(t) = \cos(3t)u(t)$
For ROC: Re{s} < 0,	$x(t) = -\cos(3t)u(-t)$

Problem 5

Find inverse Laplace transform of $X(s) = \dfrac{s+2}{s^2+7s+12}$ $\quad ROC: -4 < \mathrm{Re}\{s\} < -3$

$$X(s) = \frac{s+2}{s^2+7s+12} = \frac{s+2}{(s+4)(s+3)} = \frac{A}{s+4} + \frac{B}{s+3} = \frac{2}{s+4} - \frac{1}{s+3}$$

ILT

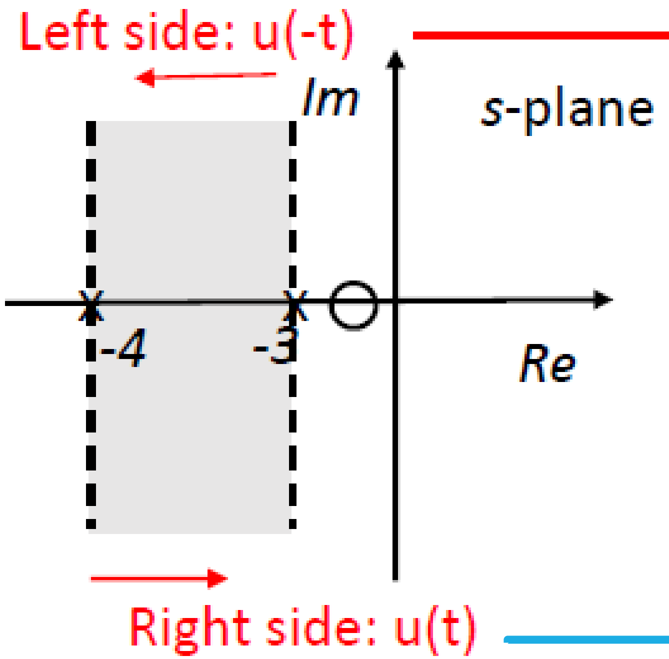
$$\Rightarrow x(t) = 2e^{-4t}u(t) - (-)e^{-3t}u(-t) = 2e^{-4t}u(t) + e^{-3t}u(-t)$$

$Re\{s\} > -4$

$$\frac{2}{s+4} \overset{\mathcal{L}}{\Longleftrightarrow} 2e^{-4t}u(t)$$

$Re\{s\} < -3$

$$\frac{-1}{s+3} \overset{\mathcal{L}}{\Longleftrightarrow} e^{-3t}u(-t)$$



s+2 = A(s+3) + B(s+4)

s+2 = (A+B)s + 3A+4B

A+B = 1

3A+4B = 2

∴ A = 2, B = -1

Problem 6

Determine $Y(s)$, when

$$y(t) = x_1(t-2) * x_2(-t+3)$$
$$x_1(t) = e^{-2t}u(t)$$
$$x_2(t) = e^{-3t}u(t)$$

$$e^{-at}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+a} \quad ROC: Re\{s\} > -a$$

$$x(t-t_0) \overset{\mathcal{L}}{\longleftrightarrow} e^{-st_0}X(s)$$

$$x(-t) \overset{\mathcal{L}}{\longleftrightarrow} X(-s)$$

$$X_1(s) = \frac{1}{s+2}; \quad ROC > -2$$

$$X_2(s) = \frac{1}{s+3}; \quad ROC > -3$$

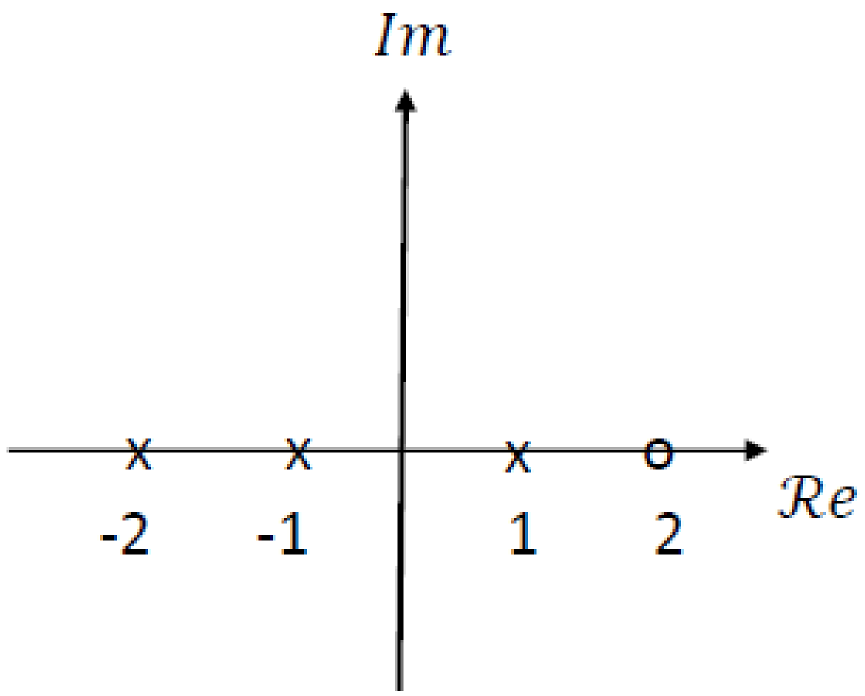
$$x_1(t-2) \overset{\mathcal{L}}{\longleftrightarrow} e^{-2s}X_1(s) = e^{-2s}\left[\frac{1}{s+2}\right] = \frac{e^{-2s}}{s+2}; \quad ROC > -2$$

$$x_2(-t+3) = x_2(-(t-3)) \overset{\mathcal{L}}{\longleftrightarrow} e^{-3s}X_2(-s) = e^{-3s}\left[\frac{1}{-s+3}\right] = \frac{e^{-3s}}{-s+3}; \quad ROC < 3$$

$$Y(s) = \left[\frac{e^{-2s}}{s+2}\right]\left[\frac{e^{-3s}}{-s+3}\right] = \frac{e^{-5s}}{(s+2)(-s+3)}$$

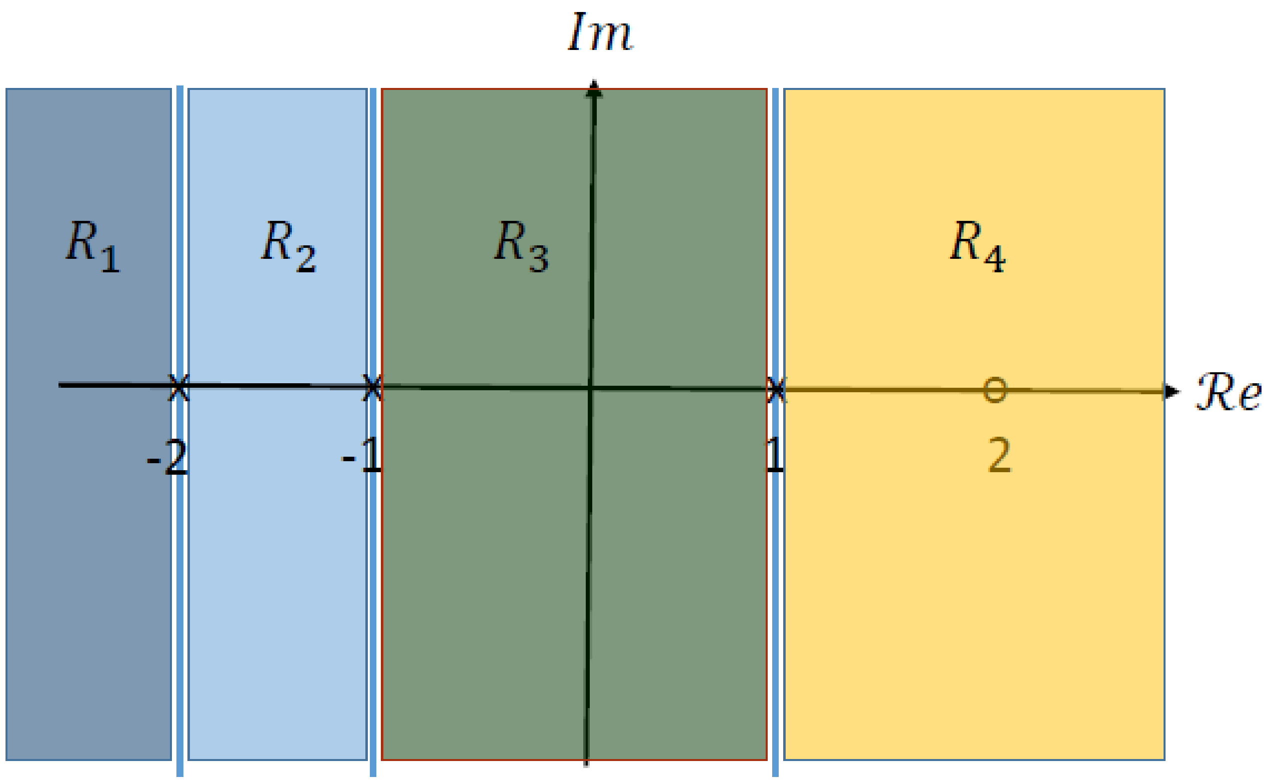
Problem 7

An LTI system $H(s)$ has pole-zero plot:



(a) Indicate all possible ROCs

(b) Specify whether the system:
Stable and/or Causal from part (a)



R_1: NOT Causal and NOT Stable ROC: Re{s} < -2

R_2: NOT Causal and NOT Stable ROC: -2 < Re{s} < -1

R_3: NOT Causal and Stable ROC: -1 < Re{s} < 1

R_4: Causal and NOT Stable ROC: Re{s} > 1

Problem 8

LTI system has differential equation: $$\frac{d^2y(t)}{dt^2} - \frac{dy(t)}{dt} - 2\,y(t) = x(t)$$

(a) Determine $H(s)$ as a ratio of two polynomials in s, and sketch the pole-zero plot.

$$\frac{d}{dt}x(t) \overset{\mathcal{L}}{\Longleftrightarrow} s\,X(s)$$

$$\frac{d^2y(t)}{dt^2} - \frac{dy(t)}{dt} - 2\,y(t) = x(t)$$

$$\overset{\mathcal{L}}{\Rightarrow} s^2Y(s) - s\,Y(s) - 2Y(s) = X(s)$$

$$\Rightarrow Y(s)[s^2 - s - 2] = X(s)$$

$$\Rightarrow H(s) = \frac{Y(s)}{X(s)} = \frac{1}{s^2 - s - 2} = \frac{1}{(s-2)(s+1)}$$

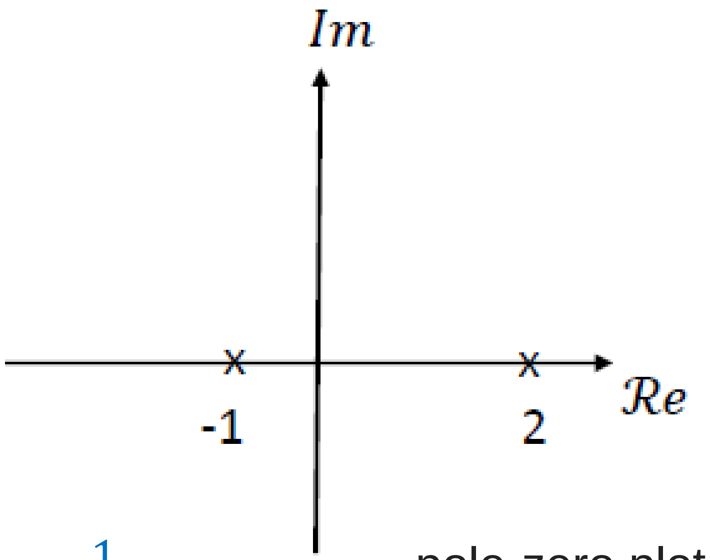
$$\Rightarrow H(s) = \frac{Y(s)}{X(s)} = \frac{A}{(s-2)} + \frac{B}{(s+1)}$$

$$\Rightarrow \frac{1}{s^2 - s - 2} = \frac{As - 2A + Bs - 2B}{s^2 - s - 2}$$

$$\Rightarrow \frac{1}{s^2 - s - 2} = \frac{(A+B)s - 2A - 2B}{s^2 - s - 2}$$

$$\Rightarrow \begin{cases} A + B = 0 \\ -2A - 2B = 1 \end{cases} \Rightarrow \begin{cases} A = \frac{1}{3} \\ B = -\frac{1}{3} \end{cases}$$

$$\Rightarrow H(s) = \frac{\frac{1}{3}}{(s-2)} - \frac{\frac{1}{3}}{(s+1)}$$



pole-zero plot

Problem 8 - conted

(b) Determine $h(t)$ for the following cases:

1: The system is Stable

2: The system is Causal

3: The system is neither Stable nor Causal

$$H(s) = \frac{\frac{1}{3}}{(s-2)} - \frac{\frac{1}{3}}{(s+1)}$$

Poles at -1, 2

- **Stable:** $-1 < ROC < 2$

  $$h_1(t) = -\frac{1}{3}e^{2t}u(-t) - \frac{1}{3}e^{-t}u(t)$$

- **Causal:** $ROC > 2$

  $$h_1(t) = \frac{1}{3}e^{2t}u(t) - \frac{1}{3}e^{-t}u(t)$$

- **Not Stable and Not Causal:** $ROC < -1$

  $$h_1(t) = -\frac{1}{3}e^{2t}u(-t) + \frac{1}{3}e^{-t}u(-t)$$