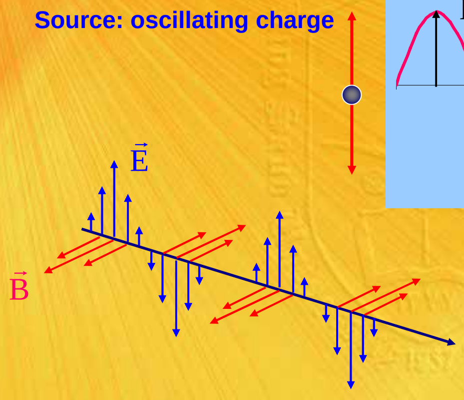
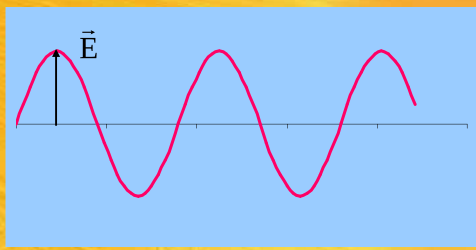
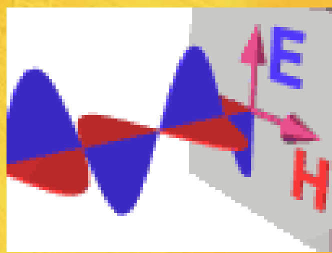
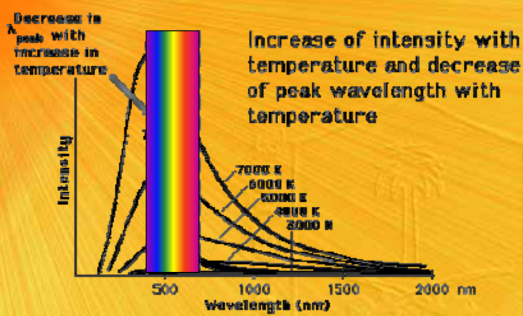
SUMMARY OF BLACKBODY RADIATION THEORY

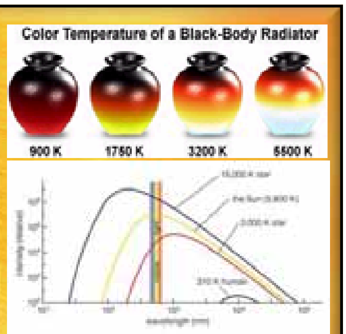
EM radiation
Source: oscillating charge











General Principles

- All bodies emit radiation continuously whatever their temperature.
- The predominant frequency ("color") depends on the temperature.
- Mostly infrared at room temperature.
- The ability of a body to radiate is closely related to its ability to absorb radiation.
- At thermal equilibrium the rate of radiation is equal to the rate of absorption of a body.

Blackbody

- Ideal body
- Absorbs all radiation regardless of the frequency

Blackbody Radiator

- Cavity with tiny hole (for example, a rectangular box)
- Filled with radiation at equilibrium
- Filled with standing waves at equilibrium with wavelengths

WIEN'S DISPLACEMENT LAW

As a body is heated the wavelength of maximum energy density is reduced and is given by $\lambda_{max}T = 2.898 \times 10^{-3}$ m K

This gives rise to the concept of colour temperature where the temperature and colour are linked

WIEN'S RADIATION LAW

Wien put forward the empirical law $\rho(T) = A\,\lambda^{-5}\,e^{-B/\lambda T}$ where A and B are experimentally determined constants

Although this function works well at short wavelengths it does not give good results at long wavelengths

Example How Hot is the Sun?

Consider the Sun as a blackbody. The peak radiation occurs at 500 nm

Use Wien's displacement law

$$\lambda_{max}\,T = 2.898 \times 10^{-3} \text{ m K}$$

Thus

$T = 2.898 \times 10^{-3}/\lambda_{max} = 2.898 \times 10^{-3}$ m K/(500 x 10^{-9} m)=5800 K

Apparent surface temperature of the Sun is about 5800K

Rayleigh-Jeans Law

- **Rayleigh calculated the number of modes of vibration in the cavity**
- **He found that the number of possible modes per unit volume in the wavelength range λ to $\lambda+d\lambda$ was given by $dn_\lambda=(8\pi/\lambda^4)\, d\lambda$**
- **Thermodynamic arguments indicated that each mode had kT of energy**
- **The energy density in wavelength range λ to $\lambda+d\lambda$ was calculated to be $\rho(T) = 8\,\pi\, k\, T\, \lambda^{-4}$**
- **This is called the Rayleigh-Jeans Law**
- **Note that at short wavelength the function blows up**
- **This is called the ULTRAVIOLET CATASTROPHE**

Planck's Radiation Law

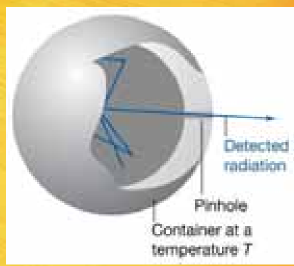
- **Max Planck (1900) used thermodynamic arguments to derive Wien's Law for short wavelengths**
- **He then used similar arguments to derive Rayleigh's Law at long wavelengths**
- **These were combined to give**
- **$\rho_\lambda=A\lambda^{-5}/(e^{B/\lambda T} - 1)$**
- **At short wavelength e $B/\lambda T >> 1$ and so expression reduces to Wien's Law**
- **At long wavelength e $B/\lambda T \sim 1 + B/\lambda T$ and so this reduces to $\rho_\lambda = (A/B\ )T\,\lambda^{-4}$ i.e. Rayleigh's Law**

Consider a cavity with walls kept at a constant temperature T.
emission from walls ➡ e.m. radiation➡ emission & absorption→ equilibrium

$$\rho = \int_0^\infty \rho_\omega d\omega = \int_0^\infty \rho_\nu d\nu \quad ; \quad where \;\; \rho_\nu \; or \; \rho_\omega = spectral\ energy\ density$$

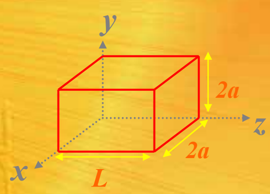
$$\rho_\nu d\nu \equiv energy\ density\ in\ the\ freq.\ range\ from\ \nu\ to\ \nu + d\nu$$

In order to comply with the 2nd law of thermodynamic,
ρ_ν (and ρ_ω) is a universal function of frequency and temperature, Not a function of the shape of the cavity



We may consider a simple rectangular cavity uniformly filled with a dielectric and having perfectly conducting walls .

First, one must count the number of waves inside a black cavity of dimension $2a \times 2a \times L$



$V=(2a)(2a)(L)$

To find ρ_ν or ρ_ω we need to know what standing wave, with what frequencies exist in the cavity. And what is the average energy.

The allowed standing waves are called NORMAL MODES

According to Maxwell's equations, the electric field $E(x,y,z,t)$ must satisfy the wave equation

$$\nabla^2 \vec{E} - \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} = 0 \qquad (1)$$

The field must be satisfy the following boundary condition at each wall;

$$\vec{E} \times \vec{n} = 0 \qquad (*)$$

Where n is the normal to the particular wall under consideration

Let $\vec{E} = \vec{u}(x,y,z)A(t)$ *(2)*

and substitute in (1), we have

$$\nabla^2 \vec{u} = -k^2 \vec{u} \qquad (3)$$

$$\frac{d^2 \vec{A}}{dt^2} = -(ck)^2 \vec{A} \qquad (4)$$

Where k is a constant. Equation (4) has the general solution

$$A = A_0 \sin(\omega t + \phi) \qquad (5)$$

Where A_0 *and* ϕ are arbitrary constants and $\omega = ck$.

The solution given by (5) corresponds to a standing wave configuration of e.m. field within the cavity.

In fact the amplitude of oscillation at a given point of the cavity is constant in time. A solution of this type is called an e.m. mode of the cavity.

Equation (3) is Helmholtz eq. , under B.C. gives the following ;

$$\left.\begin{aligned} u_x &= e_x \cos(k_x x)\sin(k_y z)\sin(k_z z) \\ u_y &= e_y \sin(k_x x)\cos(k_y z)\sin(k_z z) \\ u_z &= e_z \sin(k_x z)\sin(k_y z)\cos(k_z z) \end{aligned}\right\} \qquad (6)$$

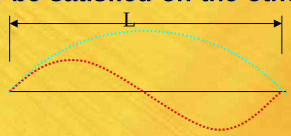
Satisfy eq.(3) for any value of e_x, e_y, e_z, provided that ;

$$k^2 = k_x^{\,2} + k_y^{\,2} + k_z^{\,2} \qquad (7)$$

The solution (6) already satisfy the B.C. (*) on the three planes x=0, y=0, z=0

If the condition that eq. (*) should also be satisfied on the other walls of the cavity, we have;

$$k_x = \frac{l\pi}{2a},\; k_y = \frac{m\pi}{2a},\; k_z = \frac{n\pi}{L}$$



l, m and n are positive integers ➨ represent the number of nodes that the standing wave modes has along x, y and z respectively

The freq. ω of the mode will be;

$$\omega^2_{l,m,n} = c^2\left[\left(\frac{l\pi}{2a}\right)^2 + \left(\frac{m\pi}{2a}\right)^2 + \left(\frac{n\pi}{L}\right)^2\right]$$

If N_ν represent the number of modes in the cavity with freq. between 0 and ν

Next thing would be to calculate the number of modes between the frequency range of (0,ν).

In wave number units this is the range (0, 2πν/c). The number of modes and the density can be calculated by treating each combination of the numbers (k_x ,k_y ,k_z) as a point in the positive octant of the Cartesian coordinate system. The unit cell of the cube is defined by the lengths (π/2a, π/2a, π/L) The number of nodes in this cube is

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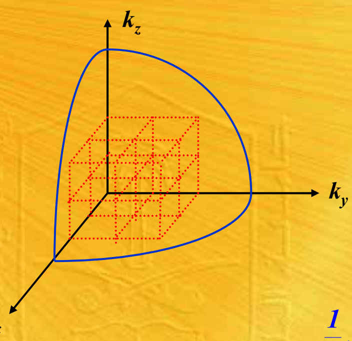

$$N_\nu = \frac{volume\ in\ positive\ octant\ in\ freq.space}{volume\ of\ one\ mode} = 2\frac{\frac{1}{8}\cdot\frac{4}{3}\pi\left(\frac{2\pi\nu}{c}\right)^3}{(\frac{\pi}{2a})(\frac{\pi}{2a})(\frac{\pi}{L})} = \frac{8\pi\nu^3}{3c^3}V$$

The factor 2 is for the two perpendicular directions of polarization

where V is the total volume of the cavity

The corresponding number of modes per unit volume and per unit frequency range is

$$p(\nu) = \frac{dN_\nu / d\nu}{V} = \frac{8\pi\nu^2}{c^3} \qquad \textit{(mode density)}$$

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Since energy density is given by;

$$\rho_\nu = p(\nu)\langle E \rangle$$

Where <E> is the average energy in each mode

By assuming a continuous spectrum (high number of modes), the average energy in a given temperature T can be obtained directly from the Boltzmann statistics;

The probability dp that the energy of a given cavity mode lie between E and E+dE is express by $dp=C\ exp[-(E/kT)]$, where C is a constant.

The average energy of the mode <E> is therefore given by;

$$\langle E \rangle = \frac{\int_0^\infty E\, e^{-E/kT} dE}{\int_0^\infty e^{-E/kT} dE} = kT$$

➨ *Each "vibration" or normal mode can take a continuous range of energy*

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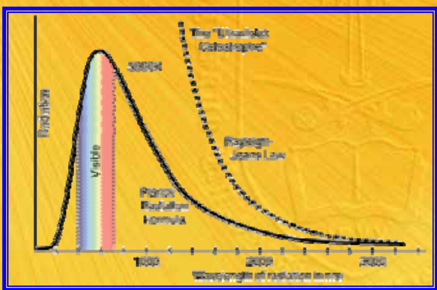
Thus we would get the energy density

$$\rho_\nu = p(\nu)\langle E \rangle = \frac{8\pi\nu^2}{c^3} kT$$

➨ *Rayleigh- Jeans Law* — WRONG

Completely disagreement with experimental results



Right away we know that the previous so called Rayleigh-Jeans equation must be wrong since there is no limit for the energy density if T $\rightarrow \infty$.

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The correction was, however, not at all an easy task to find but required totally new concepts to be taken into account. Instead of assuming totally continuous spectrum the introduction of the light quantum lead to the cutoff that was also intuitively needed.

Quantum hypothesis: ***Electromagnetic energy at frequency ν can only appear as a multiple of a step size $h\nu$.***

Let the allowed energies be E_1, E_2, E_3, ……

Then $\varepsilon_n = n\,h\,\nu$ and the relative probability of obtaining an energy E_j is $\exp(-E_j/kT)$ (i.e., a Boltzmann distribution)

$$\langle \varepsilon \rangle = \frac{1h\nu e^{-h\nu/kT} + 2h\nu e^{-2h\nu/kT} + 3h\nu e^{-3h\nu/kT} + \cdots}{1 + e^{-h\nu/kT} + e^{-2h\nu/kT} + e^{-3h\nu/kT} + \cdots}$$

$$= \frac{\sum_{1}^{\infty} nh\nu e^{-nh\nu/kT}}{\sum_{1}^{\infty} e^{-nh\nu/kT}} = \frac{h\nu}{e^{h\nu/kT} - 1}$$

How?

$$\langle E \rangle = \frac{\sum_{n=0}^{\infty} \frac{nh\nu}{kT} \exp(-nh\nu/kT)}{\sum_{n=0}^{\infty} \frac{1}{kT} \exp(-nh\nu/kT)} = \frac{kT \sum_{n=0}^{\infty} n\left(\frac{h\nu}{kT}\right) \exp[-n(h\nu/kT)]}{\sum_{n=0}^{\infty} \exp[-n(h\nu/kT)]} = \frac{kT \sum_{n=0}^{\infty} n\alpha \exp(-n\alpha)}{\sum_{n=0}^{\infty} \exp(-n\alpha)} \quad \text{where } \alpha = \frac{h\nu}{kT}$$

$$= -kT\alpha \frac{d}{d\alpha} \ln \sum_{n=0}^{\infty} \exp(-n\alpha) = -kT\alpha \frac{d}{d\alpha} \ln \frac{1}{1-\exp(-\alpha)} = kT\alpha \frac{d}{d\alpha} \ln[1-\exp(-\alpha)]$$

$$= kT\alpha \frac{\exp(-\alpha)}{1-\exp(-\alpha)} = \frac{h\nu}{\exp\alpha - 1} = \frac{h\nu}{\exp(h\nu/kT) - 1}$$

$$\langle E \rangle = \frac{h\nu}{\exp(h\nu/kT) - 1}$$

Note that for $h\nu/kT \to 0$, $\exp(h\nu/kT) \to 1 + h\nu/kT$, and therefore $\langle E \rangle \to kT$

Similarly, $\langle E \rangle \to 0$ as $h\nu/kT \to \infty$

Now the energy density also becomes different

$$\rho_\nu = p(\nu)\langle E \rangle = \frac{8\pi\nu^2}{c^3}\frac{h\nu}{e^{\frac{h\nu}{kT}} - 1}$$

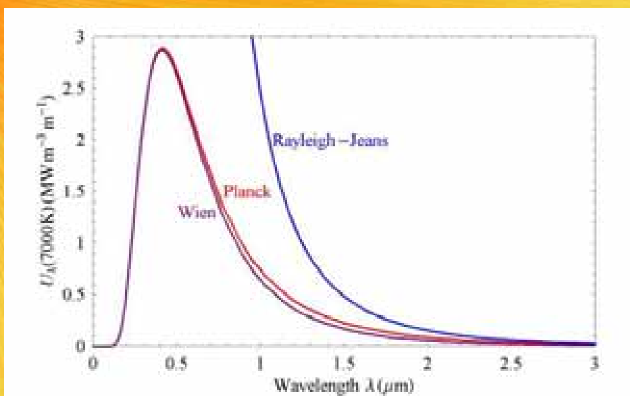
This Planck formula has also been experimentally verified

In term of ρ_ω;

$$\rho_\omega = \frac{\rho_\nu}{2\pi} = \frac{\hbar\omega^3}{\pi^2 c^3}\frac{1}{e^{\frac{\hbar\omega}{kT}} - 1}$$

Note

$$\rho_\nu = \begin{cases} \frac{4\nu^2}{c^3}kT & \text{for } h\nu/kT << 1 \\ \frac{4\nu^2}{c^3}h\nu\, e^{-h\nu/kT} & \text{for } h\nu/kT >> 1 \end{cases}$$



- Planck result fits experiment perfectly
- R-J is accurate in infrared but diverges in ultraviolet
- Wien works well except in the infrared

What's $\rho_T(\lambda)$?

We know $\nu = c/\lambda$ so $d\nu = -(c/\lambda^2)d\lambda$ and define $\rho_T(\lambda)d\lambda = -\rho_T(\nu)d\nu$

so $\rho_T(\lambda) = -\rho_T(\nu)d\nu/d\lambda = \rho_T(\nu)c/\lambda^2$

$$\rho_T(\lambda)d\lambda = \frac{8\pi h}{\lambda^3}\frac{c}{\lambda^2}\frac{d\lambda}{\exp[hc/(\lambda kT)]-1} = \frac{8\pi hc}{\lambda^5}\frac{d\lambda}{\exp[hc/(\lambda kT)]-1}$$

Note: Both Stefan's Law & Wien's displacement law can be derived from the Plank formula

One important and interesting fact is to notice that the average <u>number of photons</u> in each mode is

$$\langle q\rangle = \frac{\langle E\rangle}{h\nu} = \frac{1}{\exp(h\nu/kT)-1}$$

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<u>Exercise</u>

Determine the wavelength of maximum emission for the human body (37° C), assuming a Black-Body distribution of the emitted EM radiation.

$$\rho = \frac{dE}{d\lambda} = \frac{8\pi hc}{\lambda^5\left(e^{hc/\lambda kT}-1\right)}$$

$$h = 6.62608\cdot 10^{-34}\ J\ s \quad ,\ c = 2.99879\cdot 10^{+8}\ m\ s^{-1} \quad ,\ k = 1.38065\cdot 10^{-23}\ J\ K^{-1} \quad ,\ T = 310\ K$$

The maximum in the energy distribution is obtaining solving:

$$\frac{d\rho}{d\lambda} = 0 \Leftrightarrow \frac{d}{d\lambda}\left(\frac{8\pi hc}{\lambda^5\left(e^{hc/\lambda kT}-1\right)}\right)_{T=310} = 0$$

...

$$\lambda_{max}\cdot\left(1-\frac{1}{\exp(\frac{hc}{\lambda_{max}kT})}\right) = \frac{1}{5}\cdot\frac{hc}{kT}$$

$$If\ \frac{hc}{\lambda_{max}kT} >> 1 \Leftrightarrow \frac{hc}{kT} >> \lambda_{max}$$

$$\lambda_{max}\cdot\left(1-\frac{1}{\exp(\frac{hc}{\lambda_{max}kT})}\right) \sim \lambda_{max} = \frac{1}{5}\cdot\frac{hc}{kT} = 9.282\,\mu m$$

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Exercise

A heater filament has a radius of 2 mm and a length of 200 mm. If its surface temperature is 2000 K what is the net radiated power?

- Radiated heat from object of temperature T into surroundings with temperature T_0 is given by

$R = e\,\sigma A\,(T^4 - T_0^4)$

- Since T = 2000K and T_0 = 300 K the T^4 term will be much larger than the T_0^4 (check!) and so the rate of heat loss is

$R = e\,\sigma A\,T^4$

- Surface area of cylinder is given by

$A = 2\pi r l = 2 \text{ x } 3.14 \times (2 \text{ x } 10^{-3}\text{ m}) \times 0.2\text{m} = 2.51\times10^{-3}\text{ m}^2$

- We will assume that $e = 1$, thus

$R = 1 \times (5.67\times10^{-8}\text{ W m}^{-2}\text{ K}^{-4})(2.5\text{x}10^{-3}\text{ m}^2)(2000\text{ K})^4 = 2.27\text{ kW}$

Example: Sunlight falls at the rate of 1.4 kW/m² on the earth's surface when the sun is directly overhead. The earth's orbital radius is 1.5×10¹¹ m while the sun's radius is 7.0×10⁸ m. Find the temperature of the sun's surface.

Solution:
Intensity = Power/Area = P/A
Power = IA = RA = $(1.4\text{x}10^3\text{ W/m}^2)(4\pi)(1.5\text{x}10^{11}\text{ m})^2$
= $3.96\text{x}10^{26}$ W
total power radiated by the sun

radiation rate from sun's surface:

$$R = \frac{P}{A} = \frac{P}{4\pi r_s^2} = \frac{3.96\times10^{26}\ W}{(4\pi)(7.0\times10^8)^2} = 6.43\times10^7\ W/m^2$$

Let e=1

$$T = \left(\frac{R}{e\sigma}\right)^{1/4} = \left(\frac{6.43\times10^7}{(1)(5.67\times10^{-8}}\right)^{1/4} = 5.8\times10^3\ K$$