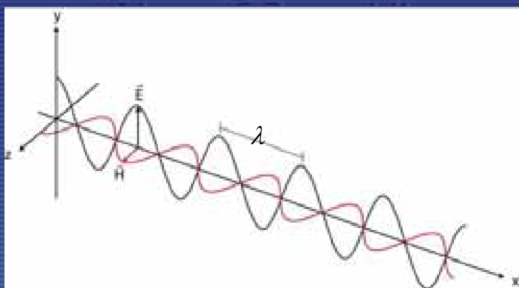
Interaction of Radiation with Matter

This lecture treats the case of the active medium interacting with optical radiation. The approximation done here is to neglect real many-body effects and take mostly atomic, ionic and molecular systems which interact weakly with other systems in the active medium. In most cases this is rather valid since lasers utilize often gases or impurity ions in solid lattices. The related material for calculating the Einstein *A* coefficient will be performed in a demo exercise.

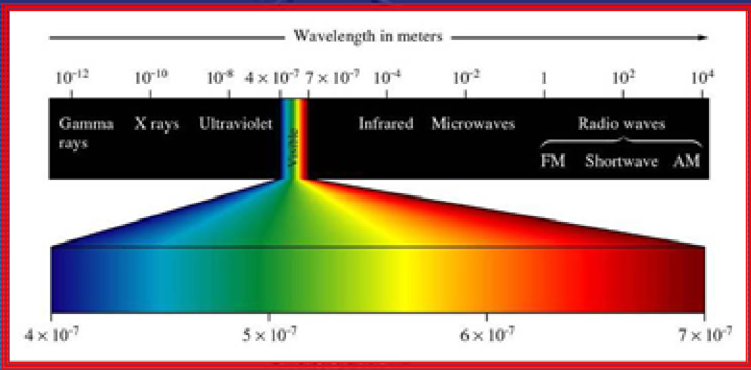
1. Plane Electromagnetic Waves- Review

Classically, the EM radiation is described as a transverse wave, consisting of an oscillating electric field and an oscillating magnetic field, mutually perpendicular.



$\lambda\nu = c$; c = speed of light in vacuum = $2.998\text{x}10^{8}$ ms^{-1}

Electromagnetic Waves Spectrum



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$$\left.\begin{matrix}E\\H\end{matrix}\right\}\ \alpha\ \frac{1}{2}\left[e^{i(\omega t-kz)}+c.c\right]$$ *two independent polarizations(transvese waves)*

Propagation of constant phase front;

ωt-kz = constant and ω ≡ angular frequency = 2πν

k ≡ wave vector = 2π/λ

$$c \equiv velocity\ of\ propagation = \frac{dz}{dt} = \frac{\omega}{k} = \frac{2\pi\nu}{k} = \omega\sqrt{\mu\varepsilon}$$

$$If\ we\ assume\ \ E(t) = E_0 \cos\omega t$$

$$\rho \equiv Energy\ Density = \frac{1}{2}\varepsilon\, E^2 + \frac{1}{2}\mu\, H^2 \xrightarrow[average]{time} 2\frac{1}{2}\varepsilon\langle E^2\rangle = \frac{1}{2}\varepsilon\, E_0^2 \quad (J/m^3)$$

$$I = int\,ensity\ or\ irradiance = c\rho = \frac{1}{2}c\varepsilon E_0^2 = \frac{(c_0/n)(\varepsilon_0 n^2)}{2}E_0^2 = \frac{1}{2}c_0\varepsilon_0 n E_0^2 \quad (W/m^2)$$

Where ; n = refractive index , ε = permittivity , μ = permeability

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2. Stationary* Quantum system vs conservative classical system

	Newton's 2nd law	Schrödinger Equation
	$F = m\frac{d^2x}{dx^2}$	$H\psi = i\hbar\frac{\partial\psi}{\partial t}$
	$E = \frac{1}{2}mv^2 + V(r)$	$H = -\frac{\hbar^2}{2m}\nabla^2 + V(r)$
Example;	$E = \frac{1}{2}mv^2 + \frac{1}{2}m\omega_0^2x^2$	$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega_0^2x^2$

For stationary system ➡ $H = H_0$ independent of time

Let $\psi(r,t) = u(r)\varphi(t)$

$$i\hbar\frac{\partial}{\partial t}[u(r)\varphi(t)] = H_0 u(r)\varphi(t) \;\Rightarrow\; \begin{cases} H_0 u(r) = Eu(r) \\ i\hbar\frac{\partial\varphi}{\partial t} = E\varphi(t) \end{cases}$$

* Is a system whose energy is constant

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There are many possible states (stationary)

$$\psi_n(r,t) = u_n(r)exp[-(iE_nt/\hbar)]$$

With stationary energy E_n

$$H_n u_n(r) = E_n u_n(r)$$

$u_n(r)$ eign – functions

E_n eign - values

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<u>**Classical**</u>

All energy are allowed

Simple harmonic oscillator for trajectory

<u>**Quantum mechanics**</u>

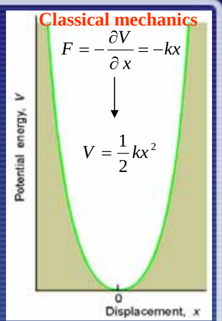
$$\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}+\frac{1}{2}kx^2\right]\psi = E\psi$$

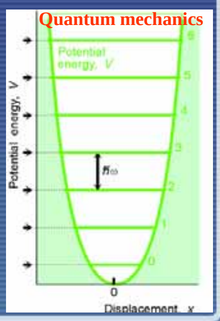
Eigen-values:

$$E_n=\left(n+\frac{1}{2}\right)\hbar\omega\ ;\quad n=0,1,2,...$$

$$\omega=\left(\frac{k}{m}\right)^{1/2}$$

- ***Energy separation*** : constant = ħω
- ***Zero-point energy*** : $E_{(v=0)}=\frac{1}{2}\hbar\omega$
- ***classical limit*** : for a huge mass m, ω is small and the energy levels form a continuum





$|u_n|^2$ Gives probability distribution of particle location

Very important, the ground state energy is $\frac{1}{2}\hbar\omega_0$, not zero

This means: even at ground state, the particle is not at rest; it move with the zero-point energy giving rise to zero-point fluctuations.

The stationary wave functions are orthogonal

$$H_0 u_n(r) = E_n u_n(r) \quad , \quad \int u_m^*(r) u_n(r) dV = \delta_{mn} \begin{cases} 1 & m=n \\ 0 & m \neq n \end{cases}$$

This means that when the system is left alone (stationary, no perturbation), stationary state are independent (i.e. no mixing)

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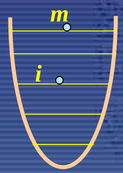
3. Time dependent perturbation theory

Suppose the system, after t=0, interacts with an external force represented by an interaction Hamiltonian H'

t < 0

$$\psi(r,t) = u_i(r) exp[-(iE_i t/\hbar)]$$



t > 0 (after interaction)

$$\psi(r,t) = \sum_{k=1}^{m} a_k \psi_k$$

$$= \sum_{k=1}^{m} a_k(t) u_k \, exp\left[-i\left(\frac{E_k}{\hbar}\right)t\right]$$

We assume the system has m stationary states

$|a_k(t)|^2$ = *probability of finding the system in state k at time t*

$$\sum_{1}^{m} |a_k(t)|^2 = 1$$

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By the time dependent Schrödinger equation;

$$H\psi = i\hbar\frac{\partial \psi}{\partial t}\ , \qquad H = H_0 + H'$$

Substitute

RHS $$i\hbar\frac{\partial \psi}{\partial t} = i\hbar\frac{\partial}{\partial t}\sum_k \left\{ a_k(t)u_k \exp\left[-i\left(\frac{E_k}{\hbar}\right)t\right]\right\}$$

$$= \sum_1^m \left(i\hbar\frac{\partial a_k}{\partial t} + E_k a_k \right) u_k e^{-iE_k t/\hbar}$$

LHS $$(H_o + H')\psi = \sum_1^m a_k(t)e^{-iE_k t/\hbar}(H_0 u_k) + \sum_1^m a_k(t)e^{-iE_k t/\hbar}(H' u_k)$$

$$H_0 u_k = E_k u_k$$

$$\therefore \quad \sum_1^m \left(i\hbar\frac{\partial a_k}{\partial t} \right) u_k e^{-iE_k t/\hbar} = \sum_1^m a_k(t)e^{-iE_k t/\hbar}(H' u_k)$$

Multiply by $u_n^*(r)e^{-iE_n t/\hbar}$ and integrate over V

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$$i\hbar\frac{\partial a_n}{\partial t} = \sum_{k=1}^m H'_{nk} a_k \exp\left[-\frac{i(E_k - E_n)}{\hbar}t\right]$$

Where; $H'_{nk}(t) = \int u_n^*(r)H'(r,t)u_k(r)dV$ (Orthogonality of u_n, u_k)

= matrix element describing the transition between n and k states.

We get m differential equations for m variables

No approximation at this point

For a m level system, there will be m equation solvable if initial conditions are known.

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For a two level system,

$$i\hbar\frac{\partial a_1}{\partial t} = H'_{11}(t)a_1 + H'_{12}(t)a_2\, e^{\frac{i(E_1-E_2)}{\hbar}t}$$

$$i\hbar\frac{\partial a_2}{\partial t} = H'_{21}(t)a_1\, e^{\frac{i(E_2-E_1)}{\hbar}t} + H'_{22}(t)a_2$$

Now approximation , we assume the interaction is weak ($H' \ll H_0$) and therefore $\left(a_2(t)\approx 0, a_1(t)\approx 1\right)$, i.e. at all time, the system is barely off thermal equilibrium – most of the time, the electron is at ground state.

$$i\hbar\frac{\partial a_1}{\partial t} = H'_{11} \approx 0$$

$$i\hbar\frac{\partial a_2}{\partial t} \approx H'_{21}\, e^{i\omega_o t} \qquad \text{Where} \quad \omega_o = \frac{(E_2 - E_1)}{\hbar}$$

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but $\quad a_2(t) \ll a_1(t)$

$a_2(t)$ is not equal to zero but it can be determine by the equation

$$i\hbar\frac{\partial a_2}{\partial t} = H'_{21}(t)\, e^{\frac{i(E_2-E_1)}{\hbar}t}$$

For Harmonic perturbation;

$$H'_{21}(t) = H'^{0}_{21}\cos\omega t = \frac{H'^{0}_{21}\left[e^{i\omega t} - e^{-i\omega t}\right]}{2i}$$

$a_2(t)$ can be obtained by direct integration

$$\frac{da_2(t)}{dt} = \frac{1}{i\hbar}\frac{H'^{0}_{21}}{2i}\left(e^{i\omega t} - e^{-i\omega t}\right)e^{i\omega_o t}$$

For $a_2(0)=0$, electron not in |2> initially

$$a_2(t) = \frac{H'^{0}_{21}}{2i\hbar}\left[\frac{\exp[i(\omega_o-\omega)t]-1}{\omega_o-\omega} - \frac{\exp[i(\omega_o+\omega)t]}{\omega_o+\omega}\right]$$

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For ω ≈ ω₀, near resonance condition, the second term can be neglected

$$a_2(t) \cong \frac{-H'^{o}_{21}}{2i}\,\frac{e^{-i\omega t}-1}{\hbar\Delta\omega} \qquad \text{where} \qquad \Delta\omega = \omega - \omega_o$$

$$|a_2(t)|^2 = \frac{|H'^{0}_{21}|^2}{\hbar}\left[\frac{\sin\left(\frac{\Delta\omega t}{2}\right)}{\Delta\omega}\right]^2$$

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Let $y = y(\Delta\omega) = \left[\frac{\sin\left(\frac{\Delta\omega t}{2}\right)}{\Delta\omega}\right]^2$

As $t \to \infty$ $y(\Delta\omega)$ becomes higher and narrower

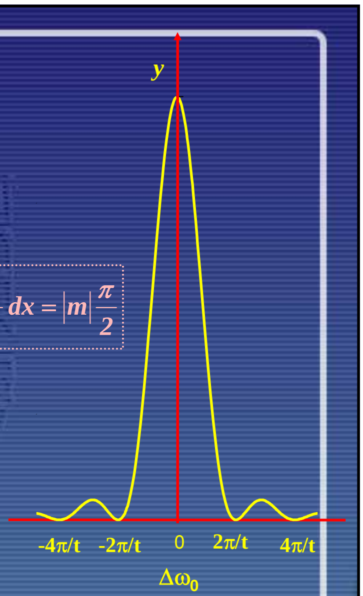
Also

$$\int_{-\infty}^{\infty} y(\Delta\omega)\,d\omega = \frac{\pi t}{2} \qquad \text{using} \quad \int_{-\infty}^{\infty}\frac{\sin^2 mx}{x^2}dx = |m|\frac{\pi}{2}$$

For large value of t ($t \to \infty$),

$$y(\Delta\omega) \cong \frac{\pi t}{2}\delta(\Delta\omega) \qquad \text{Dirac } \delta \text{ function}$$

Dirac δ - function
$f(x) = 0$
$= \infty$
and $\int_{-\infty}^{\infty} f(x)dx = 1$



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$$\therefore \; |a_2(t)|^2 = \frac{\pi |H'^{0}_{21}|^2}{2\hbar^2} t\,\delta(\Delta\omega)$$

Recall that the radiation induced transition probability from 1 ➡ 2 per unit time is designated as W_{12}.

W_{12} tells us the chance of finding the atom in level 2 per unit time.

i.e

$$\frac{d|a_2(t)|^2}{dt} = W_{12}$$

Or,

$$W_{12} = \frac{\pi |H_{21}'^{o}|^2}{2\hbar^2} \delta(\Delta\omega)$$

To calculate W_{12} explicitly, we must calculate the quantity $|H_{21}'^{o}|^2$