12: Populations in 3 and 4 level Lasers

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Generic methods of pumping

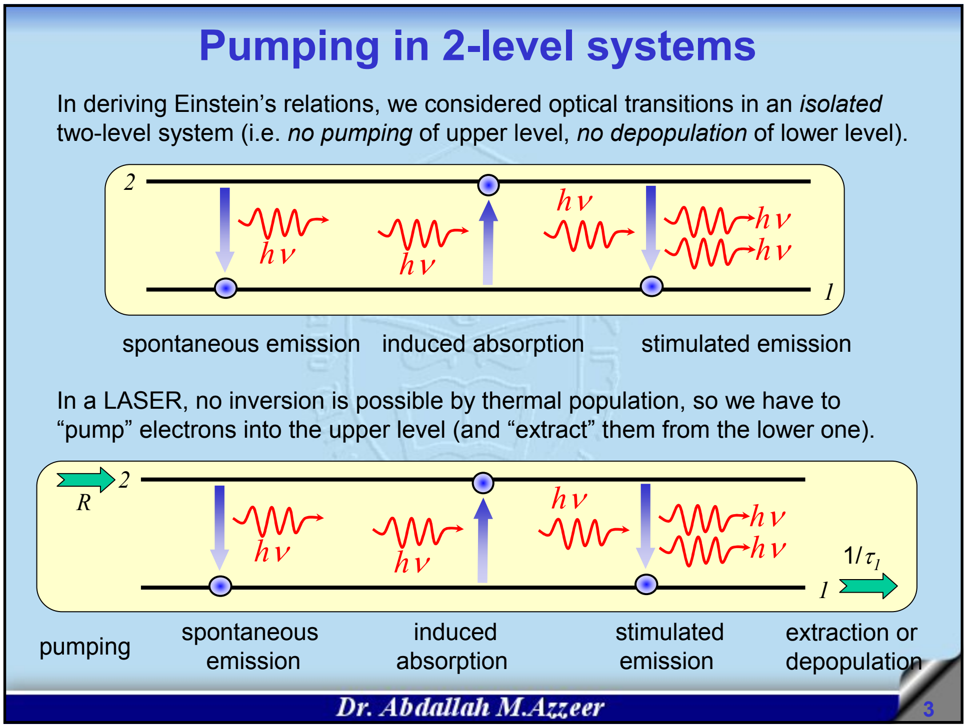
Pumping means exciting electrons from the ground state to an excited state. Excited electrons “relax”, “decay” or “cool” to the ground state by a variety of mechanisms, some of which involve the emission of light (“radiative”), and hence the possibility of stimulated emission and light amplification.

We have to provide the electrons with some energy. Common methods are:

- Electrical
  - acceleration of ions in a plasma tube
  - kinetic energy transferred to electronic transition
  - e.g. HeNe or Ar^+ laser
- Optical
  - illumination of gain medium using flashlamp
  - e.g. ruby or Nd:YAG laser
- Injection
  - direct injection of electric charge
  - e.g. semiconductor laser diode

… We will look at pumping schemes in various real lasers later on …

… in this section, we consider only *optical pumping* ...



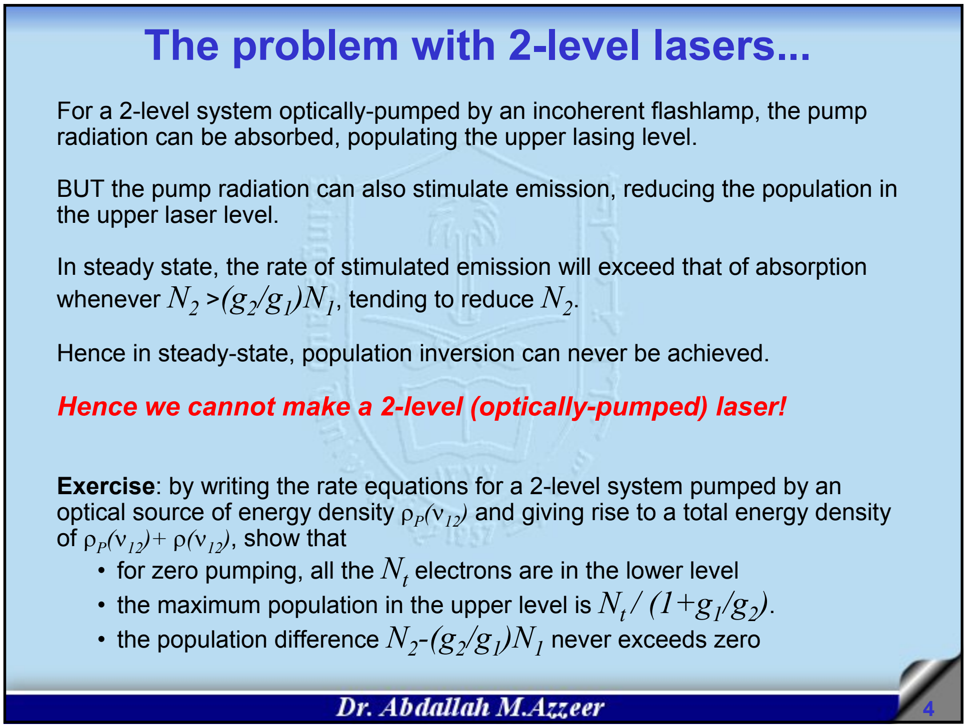
Pumping in 2-level systems

In deriving Einstein's relations, we considered optical transitions in an *isolated* two-level system (i.e. *no pumping* of upper level, *no depopulation* of lower level).

spontaneous emission induced absorption stimulated emission

In a LASER, no inversion is possible by thermal population, so we have to "pump" electrons into the upper level (and "extract" them from the lower one).

pumping spontaneous emission induced absorption stimulated emission extraction or depopulation



The problem with 2-level lasers...

For a 2-level system optically-pumped by an incoherent flashlamp, the pump radiation can be absorbed, populating the upper lasing level.

BUT the pump radiation can also stimulate emission, reducing the population in the upper laser level.

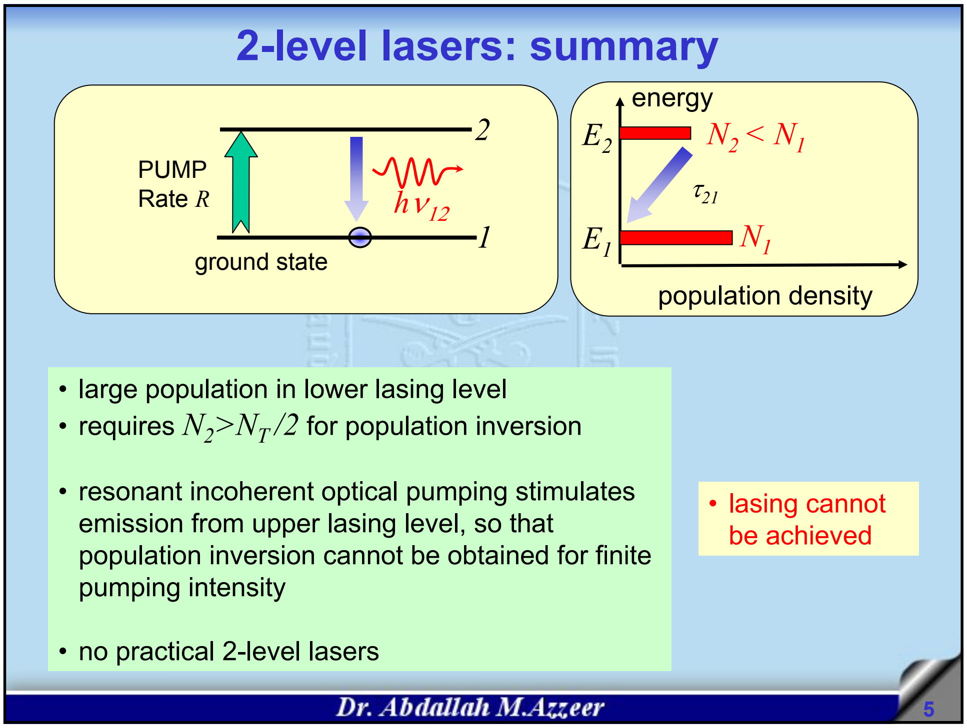
In steady state, the rate of stimulated emission will exceed that of absorption whenever $N_2 > (g_2/g_1)N_1$, tending to reduce N_2.

Hence in steady-state, population inversion can never be achieved.

Hence we cannot make a 2-level (optically-pumped) laser!

Exercise: by writing the rate equations for a 2-level system pumped by an optical source of energy density $\rho_P(\nu_{12})$ and giving rise to a total energy density of $\rho_P(\nu_{12}) + \rho(\nu_{12})$, show that

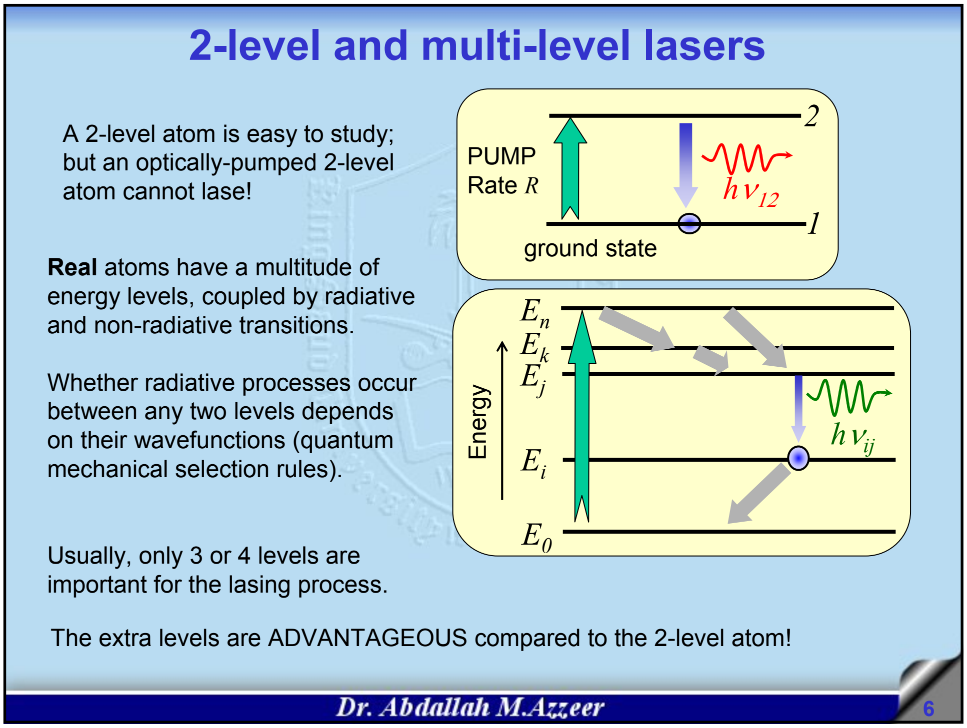
- for zero pumping, all the N_t electrons are in the lower level
- the maximum population in the upper level is $N_t / (1+g_1/g_2)$.
- the population difference $N_2-(g_2/g_1)N_1$ never exceeds zero



2-level lasers: summary

- large population in lower lasing level
- requires $N_2 > N_T/2$ for population inversion
- resonant incoherent optical pumping stimulates emission from upper lasing level, so that population inversion cannot be obtained for finite pumping intensity
- no practical 2-level lasers

- lasing cannot be achieved



2-level and multi-level lasers

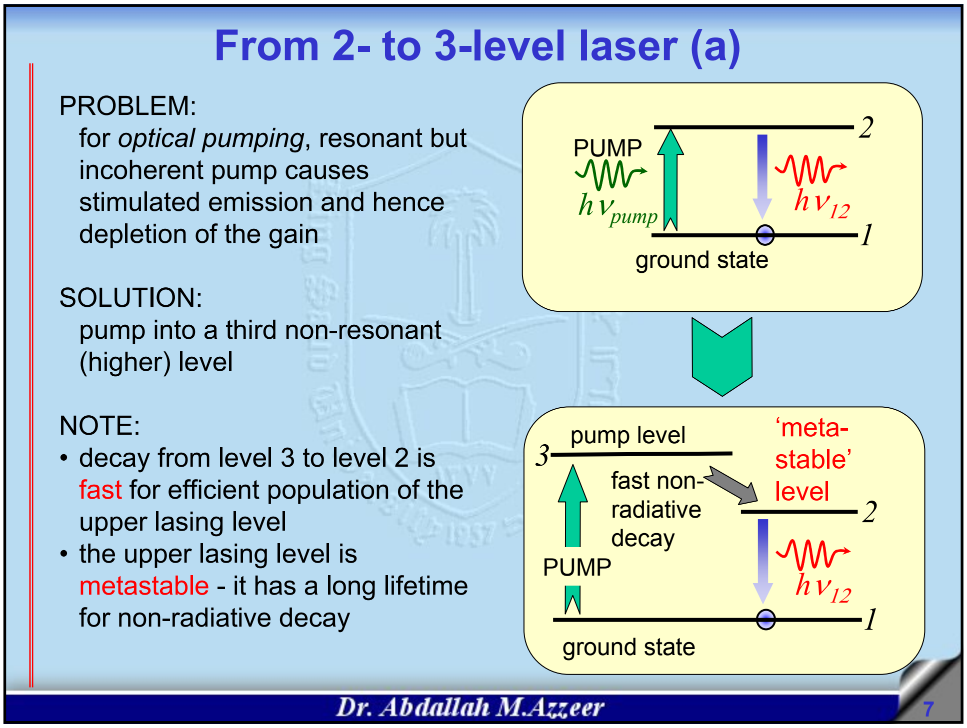
A 2-level atom is easy to study; but an optically-pumped 2-level atom cannot lase!

Real atoms have a multitude of energy levels, coupled by radiative and non-radiative transitions.

Whether radiative processes occur between any two levels depends on their wavefunctions (quantum mechanical selection rules).

Usually, only 3 or 4 levels are important for the lasing process.

The extra levels are ADVANTAGEOUS compared to the 2-level atom!

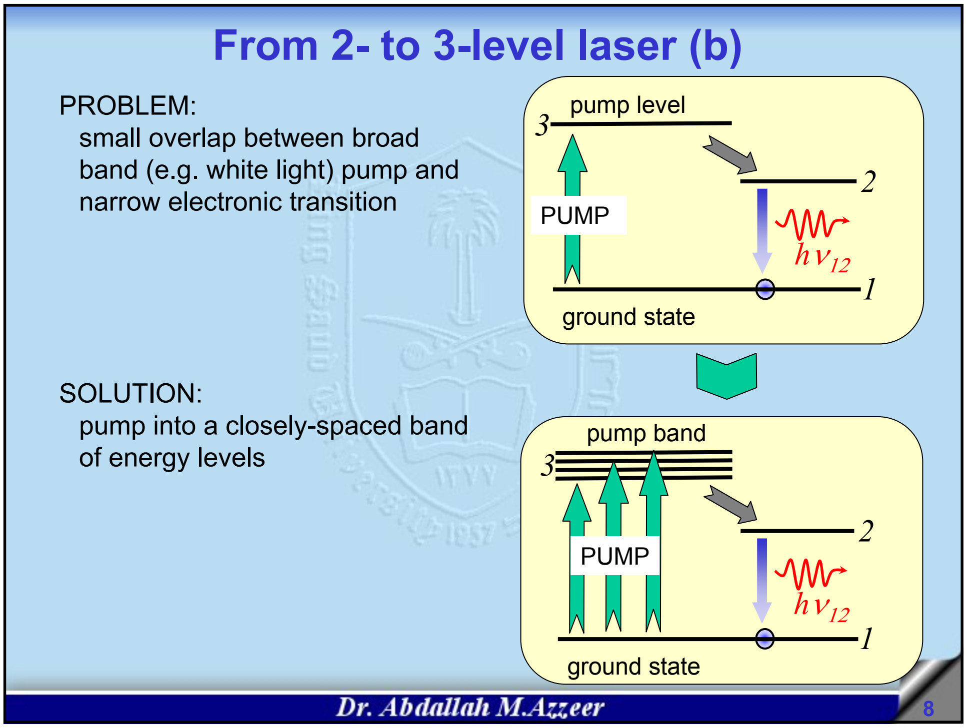


From 2- to 3-level laser (a)

PROBLEM:
for *optical pumping*, resonant but incoherent pump causes stimulated emission and hence depletion of the gain

SOLUTION:
pump into a third non-resonant (higher) level

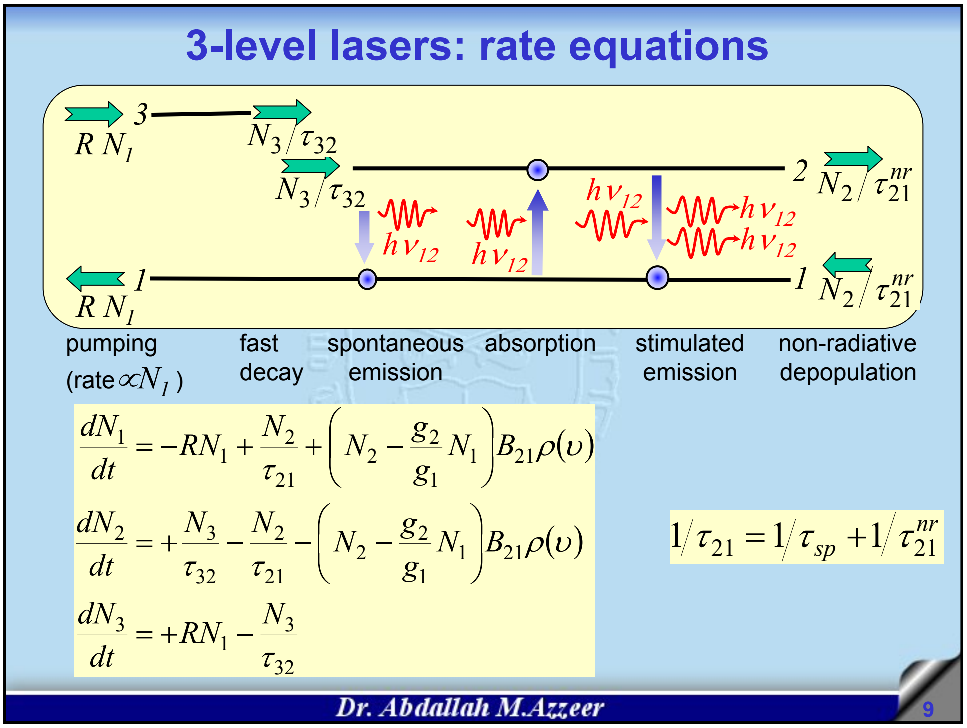
NOTE:
- decay from level 3 to level 2 is fast for efficient population of the upper lasing level
- the upper lasing level is metastable - it has a long lifetime for non-radiative decay



From 2- to 3-level laser (b)

PROBLEM:
small overlap between broad band (e.g. white light) pump and narrow electronic transition

SOLUTION:
pump into a closely-spaced band of energy levels



3-level lasers: rate equations

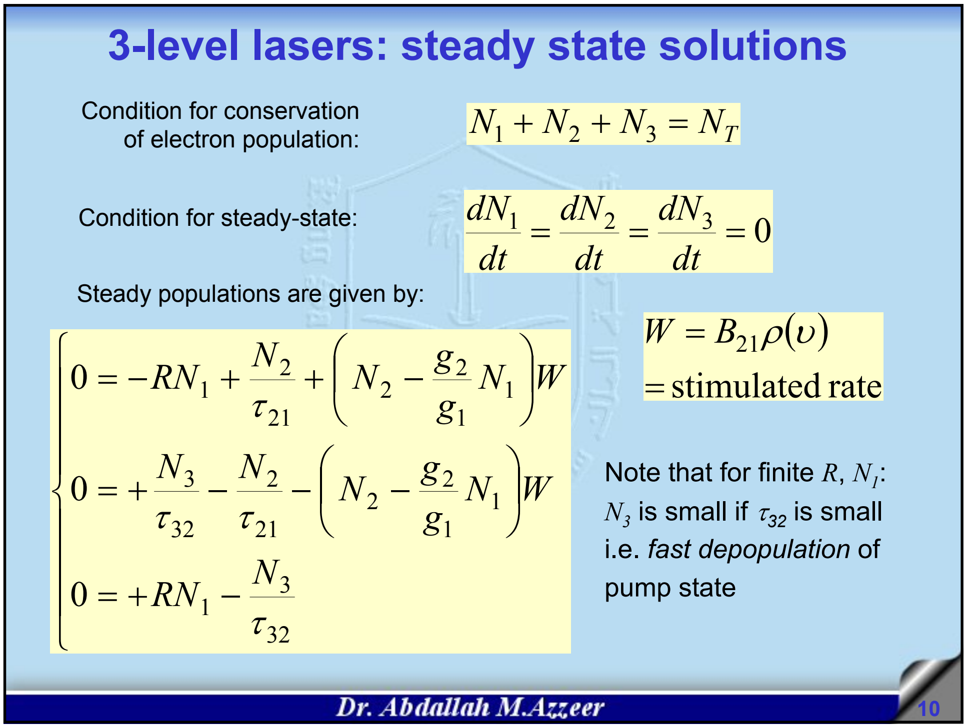
pumping (rate $\propto N_1$) | fast decay | spontaneous emission | absorption | stimulated emission | non-radiative depopulation

$$\frac{dN_1}{dt} = -RN_1 + \frac{N_2}{\tau_{21}} + \left(N_2 - \frac{g_2}{g_1}N_1\right)B_{21}\rho(\upsilon)$$

$$\frac{dN_2}{dt} = +\frac{N_3}{\tau_{32}} - \frac{N_2}{\tau_{21}} - \left(N_2 - \frac{g_2}{g_1}N_1\right)B_{21}\rho(\upsilon)$$

$$\frac{dN_3}{dt} = +RN_1 - \frac{N_3}{\tau_{32}}$$

$$1/\tau_{21} = 1/\tau_{sp} + 1/\tau_{21}^{nr}$$



3-level lasers: steady state solutions

Condition for conservation of electron population: $N_1 + N_2 + N_3 = N_T$

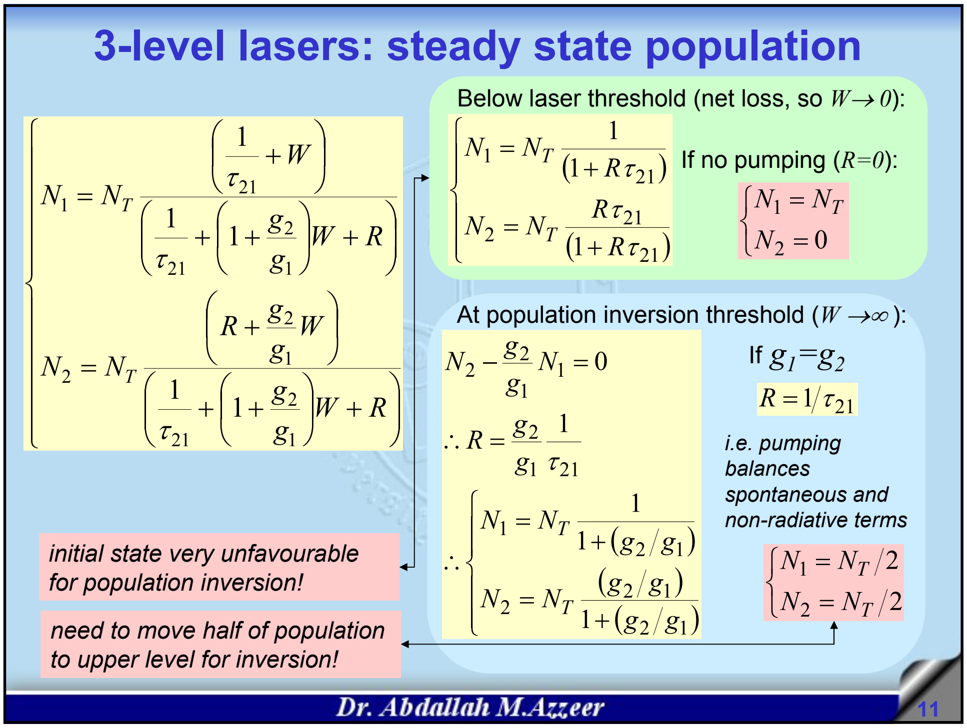
Condition for steady-state: $\frac{dN_1}{dt} = \frac{dN_2}{dt} = \frac{dN_3}{dt} = 0$

Steady populations are given by:

$$\begin{cases} 0 = -RN_1 + \dfrac{N_2}{\tau_{21}} + \left(N_2 - \dfrac{g_2}{g_1}N_1\right)W \\ 0 = +\dfrac{N_3}{\tau_{32}} - \dfrac{N_2}{\tau_{21}} - \left(N_2 - \dfrac{g_2}{g_1}N_1\right)W \\ 0 = +RN_1 - \dfrac{N_3}{\tau_{32}} \end{cases}$$

$W = B_{21}\rho(\upsilon)$
$= \text{stimulated rate}$

Note that for finite R, N_1: N_3 is small if τ_{32} is small i.e. *fast depopulation* of pump state



3-level lasers: steady state population

$$\begin{cases} N_1 = N_T \dfrac{\left(\dfrac{1}{\tau_{21}} + W\right)}{\left(\dfrac{1}{\tau_{21}} + \left(1 + \dfrac{g_2}{g_1}\right)W + R\right)} \\ N_2 = N_T \dfrac{\left(R + \dfrac{g_2}{g_1}W\right)}{\left(\dfrac{1}{\tau_{21}} + \left(1 + \dfrac{g_2}{g_1}\right)W + R\right)} \end{cases}$$

Below laser threshold (net loss, so $W \to 0$):

$$\begin{cases} N_1 = N_T \dfrac{1}{(1 + R\tau_{21})} \\ N_2 = N_T \dfrac{R\tau_{21}}{(1 + R\tau_{21})} \end{cases}$$

If no pumping ($R=0$):

$$\begin{cases} N_1 = N_T \\ N_2 = 0 \end{cases}$$

initial state very unfavourable for population inversion!

At population inversion threshold ($W \to \infty$):

$$N_2 - \frac{g_2}{g_1} N_1 = 0$$

$$\therefore R = \frac{g_2}{g_1}\frac{1}{\tau_{21}}$$

$$\therefore \begin{cases} N_1 = N_T \dfrac{1}{1 + (g_2/g_1)} \\ N_2 = N_T \dfrac{(g_2/g_1)}{1 + (g_2/g_1)} \end{cases}$$

If $g_1 = g_2$

$$R = 1/\tau_{21}$$

i.e. pumping balances spontaneous and non-radiative terms

$$\begin{cases} N_1 = N_T/2 \\ N_2 = N_T/2 \end{cases}$$

need to move half of population to upper level for inversion!

Dr. Abdallah M.Azzeer

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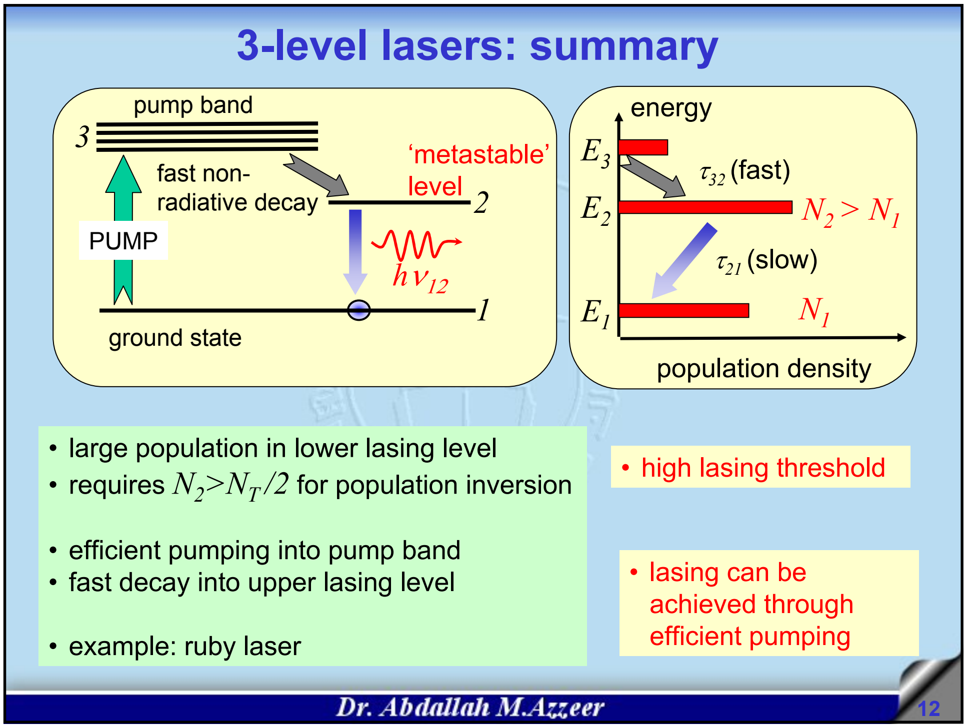

3-level lasers: summary

- large population in lower lasing level
- requires $N_2 > N_T/2$ for population inversion
- high lasing threshold

- efficient pumping into pump band
- fast decay into upper lasing level
- lasing can be achieved through efficient pumping

- example: ruby laser

Dr. Abdallah M.Azzeer

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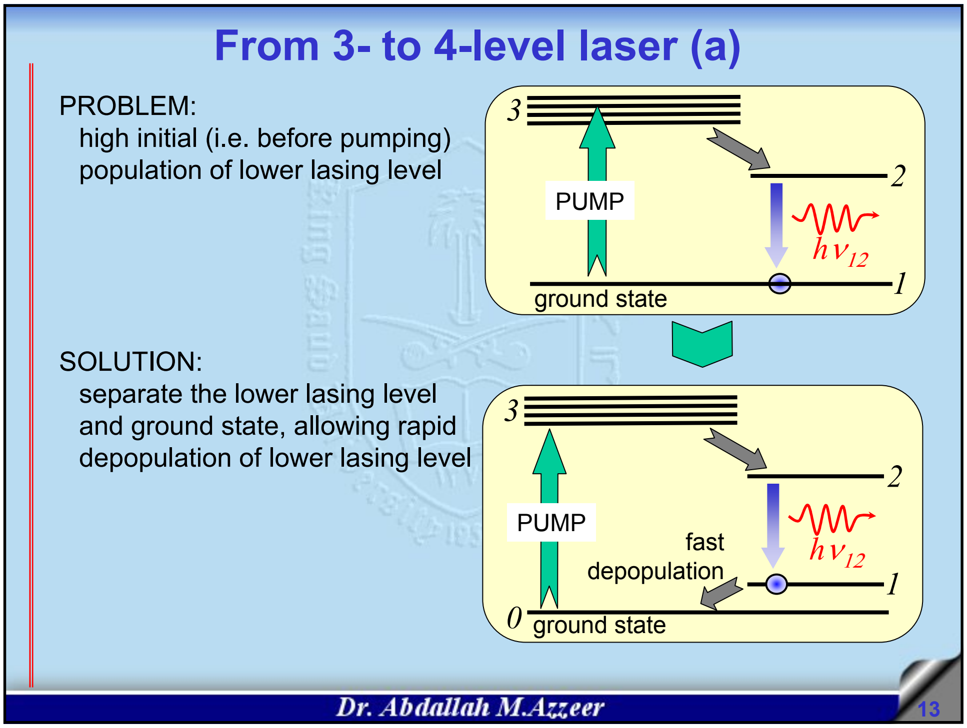
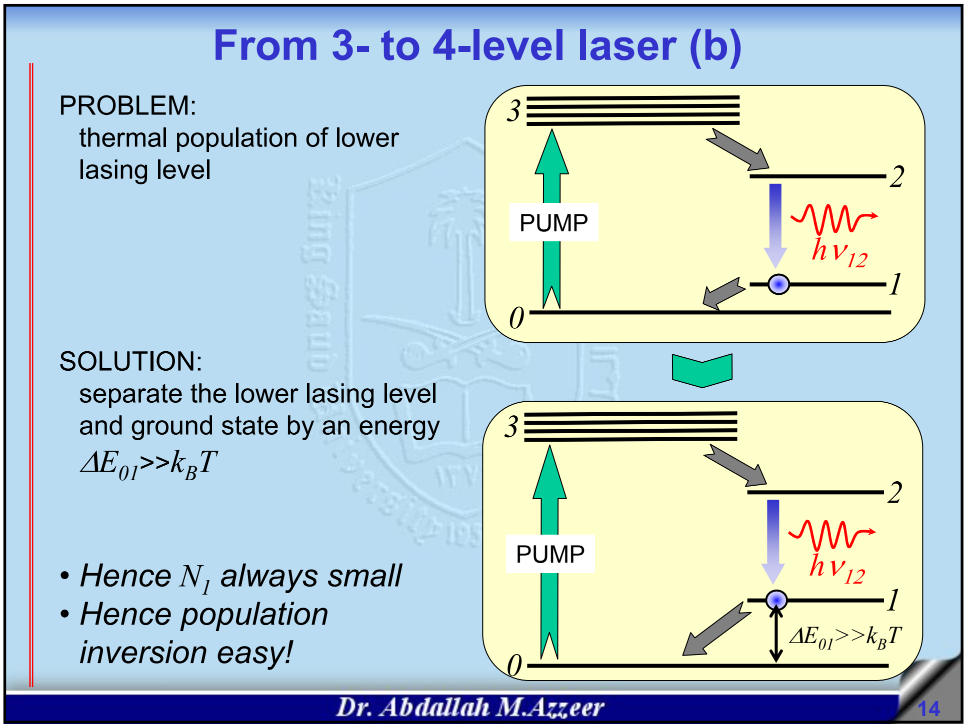


From 3- to 4-level laser (a)

PROBLEM:
high initial (i.e. before pumping) population of lower lasing level

SOLUTION:
separate the lower lasing level and ground state, allowing rapid depopulation of lower lasing level

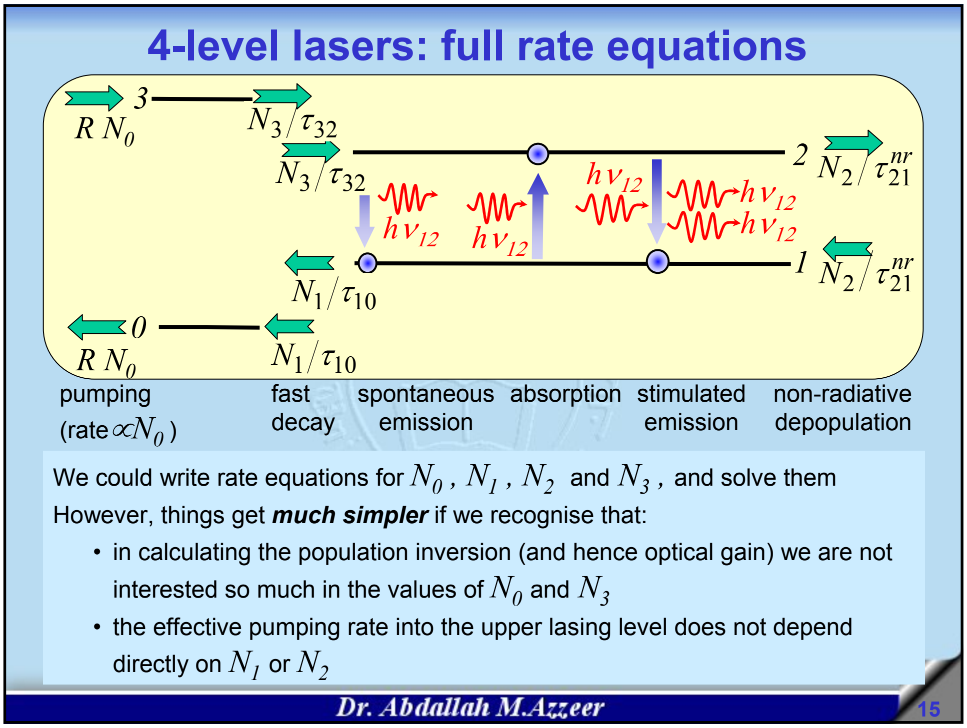


From 3- to 4-level laser (b)

PROBLEM:
thermal population of lower lasing level

SOLUTION:
separate the lower lasing level and ground state by an energy $\Delta E_{01}>>k_BT$

- *Hence N_1 always small*
- *Hence population inversion easy!*



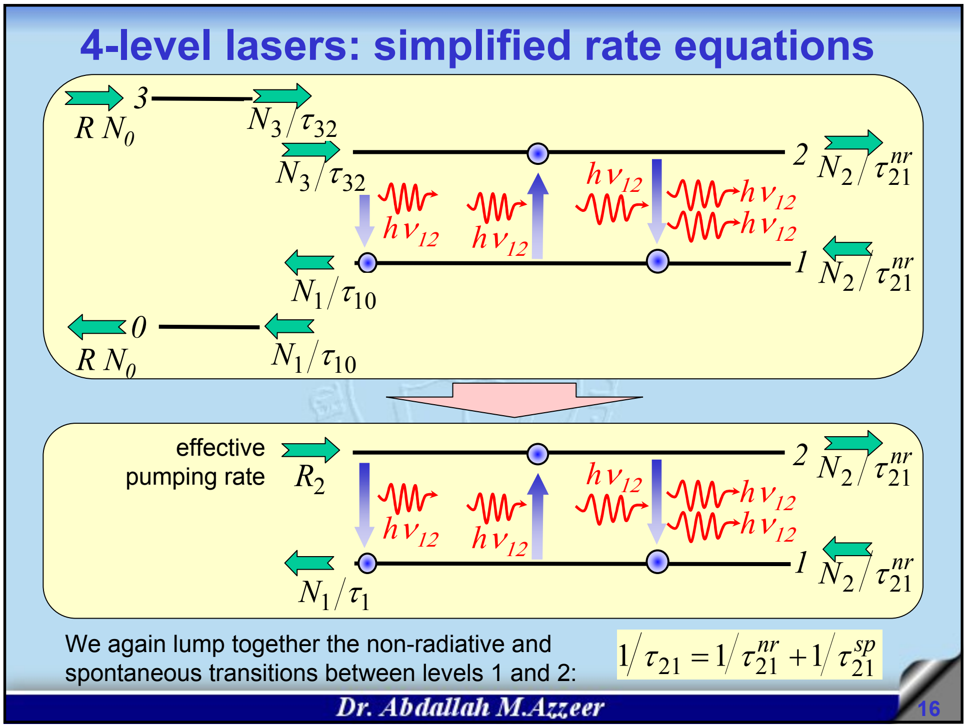
4-level lasers: full rate equations

pumping (rate $\propto N_0$) | fast decay | spontaneous emission | absorption | stimulated emission | non-radiative depopulation

We could write rate equations for N_0, N_1, N_2 and N_3, and solve them

However, things get ***much simpler*** if we recognise that:

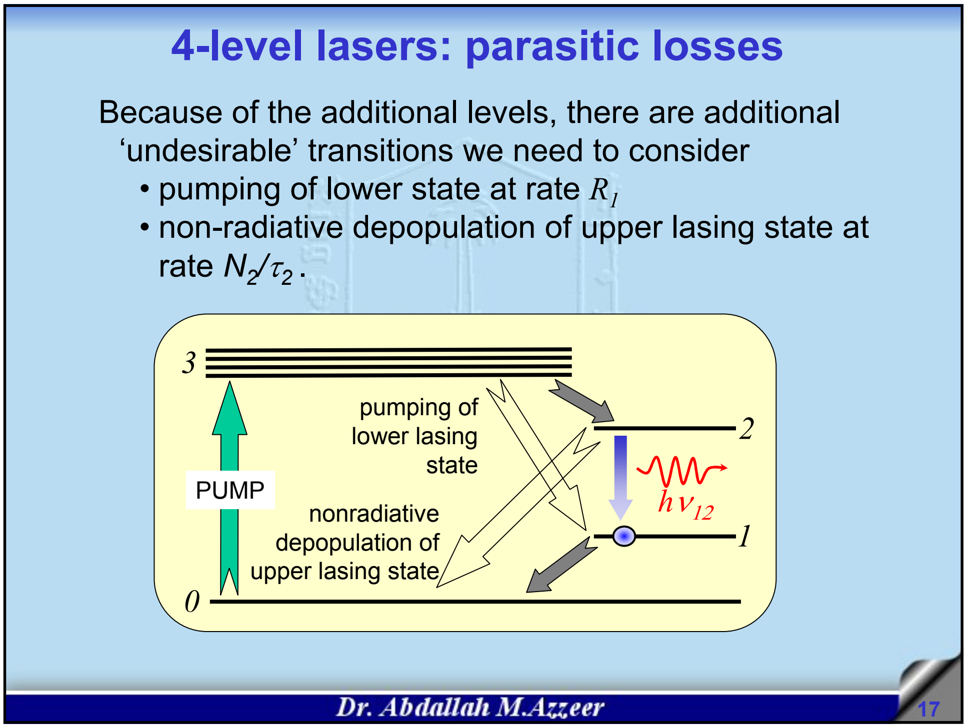
- in calculating the population inversion (and hence optical gain) we are not interested so much in the values of N_0 and N_3
- the effective pumping rate into the upper lasing level does not depend directly on N_1 or N_2



4-level lasers: simplified rate equations

We again lump together the non-radiative and spontaneous transitions between levels 1 and 2:

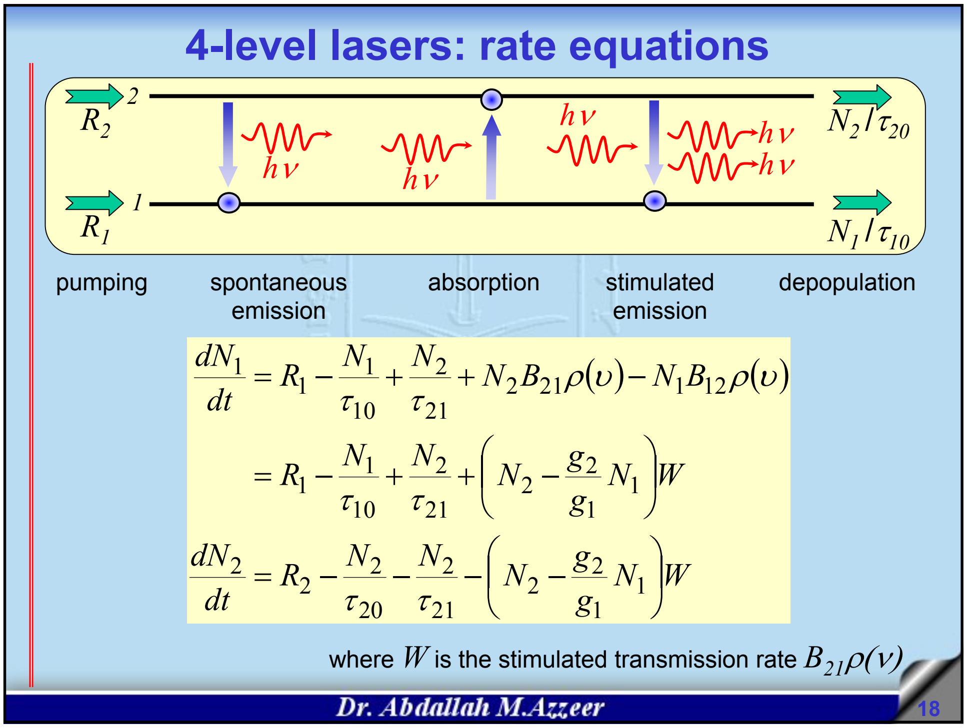
$$1/\tau_{21} = 1/\tau_{21}^{nr} + 1/\tau_{21}^{sp}$$



4-level lasers: parasitic losses

Because of the additional levels, there are additional 'undesirable' transitions we need to consider

- pumping of lower state at rate R_1
- non-radiative depopulation of upper lasing state at rate N_2/τ_2.



4-level lasers: rate equations

$$\frac{dN_1}{dt} = R_1 - \frac{N_1}{\tau_{10}} + \frac{N_2}{\tau_{21}} + N_2 B_{21}\rho(\upsilon) - N_1 B_{12}\rho(\upsilon)$$

$$= R_1 - \frac{N_1}{\tau_{10}} + \frac{N_2}{\tau_{21}} + \left(N_2 - \frac{g_2}{g_1}N_1\right)W$$

$$\frac{dN_2}{dt} = R_2 - \frac{N_2}{\tau_{20}} - \frac{N_2}{\tau_{21}} - \left(N_2 - \frac{g_2}{g_1}N_1\right)W$$

where W is the stimulated transmission rate $B_{21}\rho(\nu)$

4-level lasers: steady state equations

Condition for steady-state:

$$\frac{dN_1}{dt}=\frac{dN_2}{dt}=0$$

$$\therefore\begin{cases}0=R_1-\dfrac{N_1}{\tau_{10}}+\dfrac{N_2}{\tau_{21}}+\left(N_2-\dfrac{g_2}{g_1}N_1\right)W\\[2ex] 0=R_2-\dfrac{N_2}{\tau_{20}}-\dfrac{N_2}{\tau_{21}}-\left(N_2-\dfrac{g_2}{g_1}N_1\right)W\end{cases}$$

Rewrite and solve for:

population inversion $N_2-\frac{g_2}{g_1}N_1$

and

lower level depopulation rate $\frac{N_1}{\tau_{10}}$

$$\begin{cases}N_2-\dfrac{g_2}{g_1}N_1=\dfrac{-R_1+\dfrac{N_1}{\tau_{10}}\left(1-\dfrac{g_2}{g_1}\dfrac{\tau_{10}}{\tau_{21}}\right)}{\left(\dfrac{1}{\tau_{21}}+W\right)}\\[4ex] N_2-\dfrac{g_2}{g_1}N_1=\dfrac{R_2-\dfrac{N_1}{\tau_{10}}\dfrac{g_2\tau_{10}}{g_1}\left(\dfrac{1}{\tau_{21}}+\dfrac{1}{\tau_{20}}\right)}{\left(\dfrac{1}{\tau_{21}}+\dfrac{1}{\tau_{20}}+W\right)}\end{cases}$$

4-level lasers: lower level depopulation

Solution is:

$$\frac{N_1}{\tau_{10}}=\frac{R_1\left\{\dfrac{1}{\tau_{21}}+\dfrac{1}{\tau_{20}}+W\right\}+R_2\left\{\dfrac{1}{\tau_{21}}+W\right\}}{\left(\dfrac{1}{\tau_{21}}+\dfrac{1}{\tau_{20}}\right)+\left\{1+\dfrac{g_2}{g_1}\dfrac{\tau_{10}}{\tau_{20}}\right\}W}$$

N_1 is ALWAYS SMALL if τ_{10} is small

⇒ easy to achieve population inversion

When W is small (below lasing threshold):

$$\frac{N_1}{\tau_{10}}=\frac{R_1\left\{\dfrac{1}{\tau_{21}}+\dfrac{1}{\tau_{20}}\right\}+R_2\left\{\dfrac{1}{\tau_{21}}\right\}}{\left(\dfrac{1}{\tau_{21}}+\dfrac{1}{\tau_{20}}\right)}$$

When $(1/\tau_{20})$ is small (negligable non-radiative depopulation of upper level):

$$\frac{N_1}{\tau_{10}}=R_1+R_2$$

net pumping rate = net depopulation rate

4-level lasers: population inversion

$$N_2 - \frac{g_2}{g_1}N_1 = \frac{-R_1\left\{\frac{g_2}{g_1}\frac{\tau_{10}}{\tau_{21}}\right\}\left[\frac{1}{\tau_{20}}\left(1+\frac{\tau_{20}}{\tau_{21}}\right)+W\right]+R_2\left(1-\frac{g_2}{g_1}\frac{\tau_{10}}{\tau_{21}}\right)\left[\frac{1}{\tau_{21}}+W\right]}{\left\{\frac{1}{\tau_{21}}+W\right\}\left[\left(\frac{1}{\tau_{21}}\right)+\left(\frac{1}{\tau_{20}}\right)+\left\{1+\frac{g_2}{g_1}\frac{\tau_{10}}{\tau_{20}}\right\}W\right]}$$

Simplifies when non-radiative depopulation of upper lasing level is small:

$$\begin{cases}\tau_{20} >> \tau_{21}\\ \tau_{20} >> \tau_{10}\end{cases}$$

$$\boxed{N_2 - \frac{g_2}{g_1}N_1 = \frac{R_2\left\{1-\frac{g_2}{g_1}\frac{\tau_{10}}{\tau_{21}}\left(1+\frac{R_1}{R_2}\right)\right\}}{\frac{1}{\tau_{21}}+W} = \frac{R}{1/\tau_{21}+W}}$$

$$\text{population inversion} = \frac{\text{effective pumping rate}}{\text{rate of spontaneous + stimulated (+nonradiative) emission}}$$

For effective pumping: $R_1 << R_2$

If g_1=g_2 and τ_{21}= τ_{sp}

For population inversion: $\tau_{10} < (g_1/g_2)\tau_{21}$ ➡ $\tau_{10} < \tau_{sp}$

4-level lasers: gain saturation

$$\boxed{N_2 - \frac{g_2}{g_1}N_1 = \frac{R}{1/\tau_{21}+W}}$$

This expression is very significant: the population inversion, and hence gain coefficient γ, depends on the optical energy density W=$B_{21}\rho(\nu)$

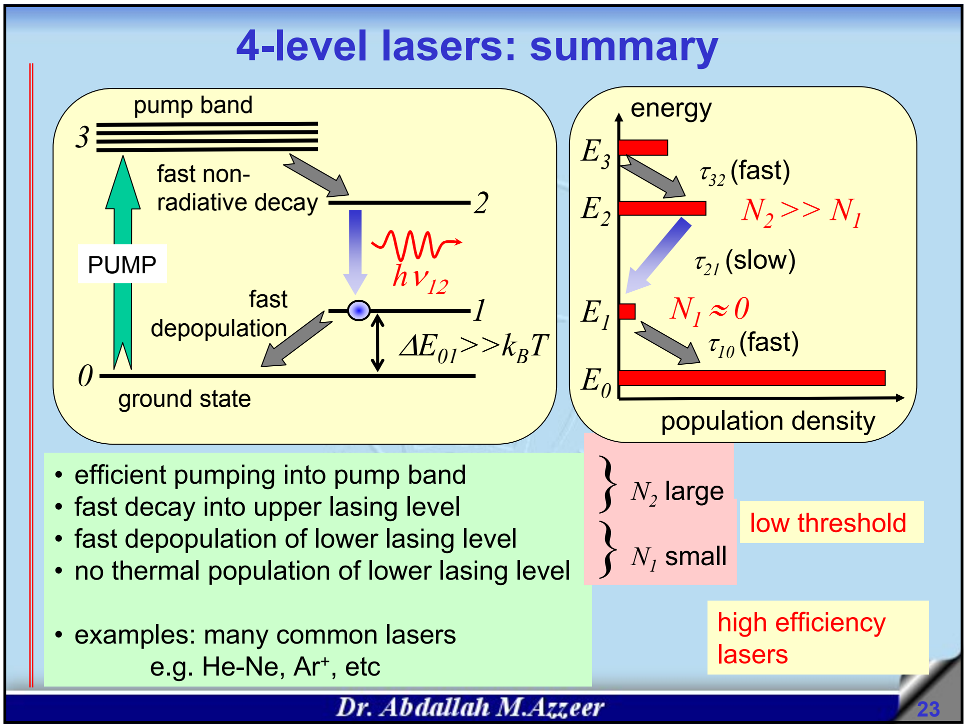
$$\gamma = \sigma_0\left(N_2 - \frac{g_2}{g_1}N_1\right) = \frac{R\sigma_0\tau_{21}}{1+\mathrm{B}_{21}\tau_{21}\rho(\nu)} = \frac{R\sigma_0\tau_{sp}}{1+\frac{c^3}{8\pi h\nu^3}\rho(\nu)}$$

When the optical energy density is zero, γ takes the value γ_0=$R\sigma_0\tau_{21}$, and it falls to half this value at the saturation density ρ_s=$(B_{21}\tau_{21})^{-1}$

$$\boxed{\gamma(\nu) = \frac{\gamma_0(\nu)}{1+\rho(\nu)/\rho_s(\nu)}}$$

Similar expressions can be written in terms of optical power and intensity.

Gain saturation is important in determining the wavelength and power of a laser…



4-level lasers: summary

- efficient pumping into pump band
- fast decay into upper lasing level
- fast depopulation of lower lasing level
- no thermal population of lower lasing level

- examples: many common lasers
  e.g. He-Ne, Ar^+, etc

} N_2 large
} N_1 small

low threshold

high efficiency lasers

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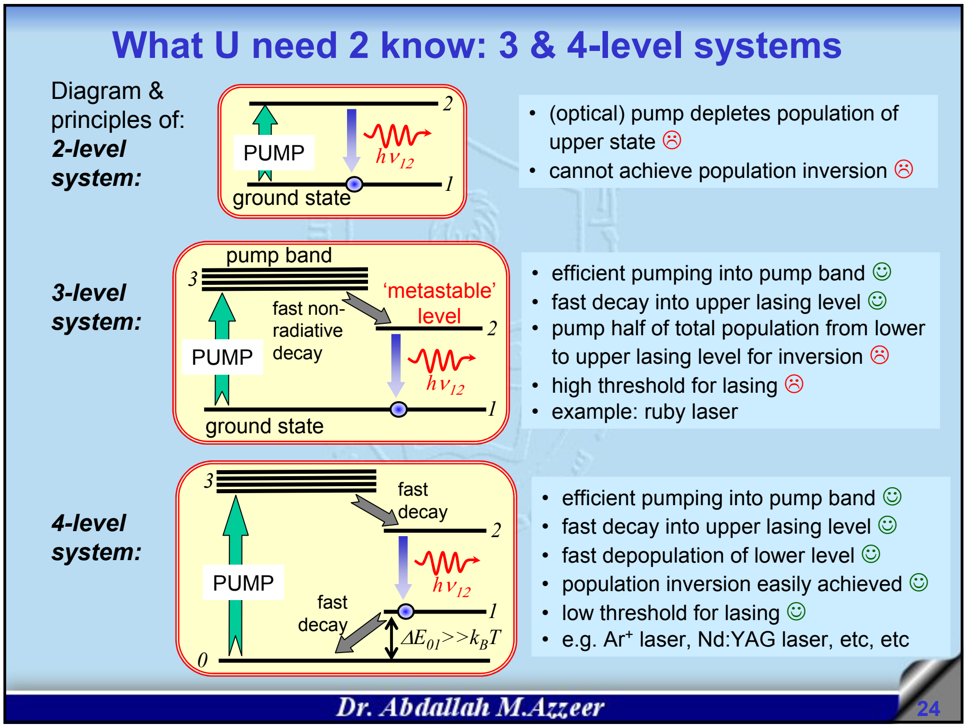

What U need 2 know: 3 & 4-level systems

Diagram & principles of:
2-level system:

- (optical) pump depletes population of upper state ☹
- cannot achieve population inversion ☹

3-level system:

- efficient pumping into pump band ☺
- fast decay into upper lasing level ☺
- pump half of total population from lower to upper lasing level for inversion ☹
- high threshold for lasing ☹
- example: ruby laser

4-level system:

- efficient pumping into pump band ☺
- fast decay into upper lasing level ☺
- fast depopulation of lower level ☺
- population inversion easily achieved ☺
- low threshold for lasing ☺
- e.g. Ar^+ laser, Nd:YAG laser, etc, etc

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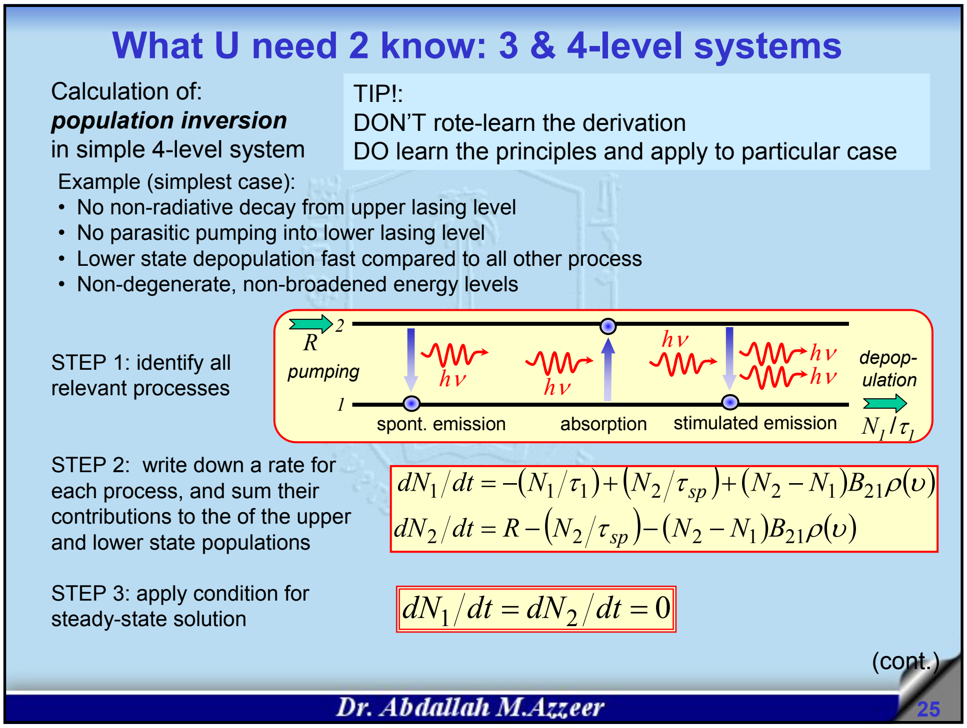


What U need 2 know: 3 & 4-level systems

Calculation of:
population inversion
in simple 4-level system

TIP!:
DON'T rote-learn the derivation
DO learn the principles and apply to particular case

Example (simplest case):
- No non-radiative decay from upper lasing level
- No parasitic pumping into lower lasing level
- Lower state depopulation fast compared to all other process
- Non-degenerate, non-broadened energy levels

STEP 1: identify all relevant processes

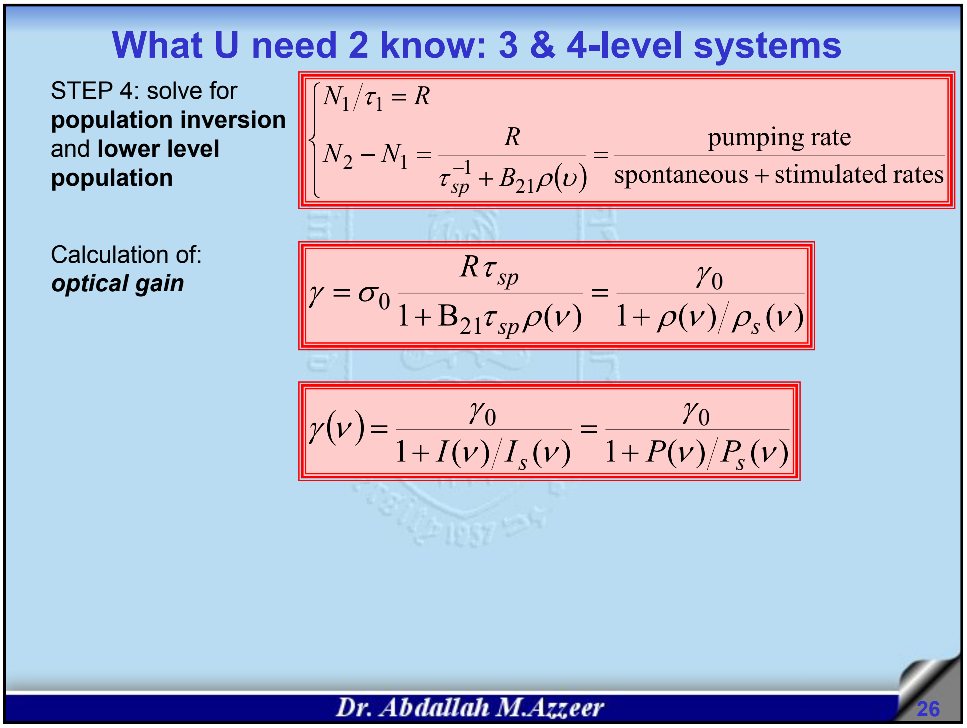
STEP 2: write down a rate for each process, and sum their contributions to the of the upper and lower state populations

$$dN_1/dt = -(N_1/\tau_1) + (N_2/\tau_{sp}) + (N_2 - N_1)B_{21}\rho(\upsilon)$$
$$dN_2/dt = R - (N_2/\tau_{sp}) - (N_2 - N_1)B_{21}\rho(\upsilon)$$

STEP 3: apply condition for steady-state solution

$$dN_1/dt = dN_2/dt = 0$$

(cont.)



What U need 2 know: 3 & 4-level systems

STEP 4: solve for **population inversion** and **lower level population**

$$\begin{cases} N_1/\tau_1 = R \\ N_2 - N_1 = \dfrac{R}{\tau_{sp}^{-1} + B_{21}\rho(\upsilon)} = \dfrac{\text{pumping rate}}{\text{spontaneous} + \text{stimulated rates}} \end{cases}$$

Calculation of:
optical gain

$$\gamma = \sigma_0 \frac{R\tau_{sp}}{1 + \mathrm{B}_{21}\tau_{sp}\rho(\nu)} = \frac{\gamma_0}{1 + \rho(\nu)/\rho_s(\nu)}$$

$$\gamma(\nu) = \frac{\gamma_0}{1 + I(\nu)/I_s(\nu)} = \frac{\gamma_0}{1 + P(\nu)/P_s(\nu)}$$