13: LASER Oscillation

CONTENTS:

- LASER optical amplication
- Optical feedback and LASER oscillation
- Cavity Lifetime
- Rate equation of laser oscillator
- Lasing threshold and gain clamping
- Optical gain and emitter power
- Dependence of output power on mirror reflectivity

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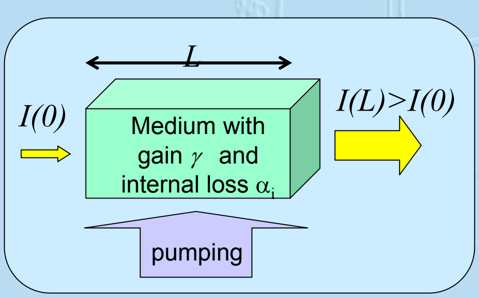
LASER Optical Amplification

LASER: Light **AMPLIFICATION** by Stimulated Emission of Radiation

First we looked at the propagation of light in a material exhibiting optical gain:



Optical Gain

$$\gamma(\nu)=\left(N_2-\frac{g_2}{g_1}N_1\right)\frac{c^2A_{21}}{8\pi\nu^2}$$

+ Pumping

$$N_2-\frac{g_2}{g_1}N_1=\frac{R}{1/\tau_{21}+W}$$

→ exponential growth (when gain exceeds internal losses)

$$I(z)=I_0\exp[(\gamma-\alpha)z]$$

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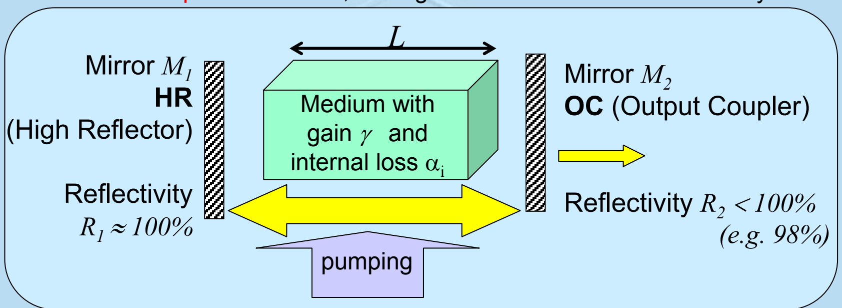
Optical Feedback and LASER Oscillation

LOSER: Light **OSCILLATOR** using Stimulated Emission of Radiation

Now we look at optical feedback, i.e. a gain material inside a laser cavity:



Why a resonator / cavity?

- γ is typically small (~0.1m^{-1}) so **multiple passes** needed for sufficient amplification (exception e.g. Nd-glass lasers)
- **optical feedback** gives self-sustained laser **oscillation** rather than laser **amplification** - and hence **longitudinal coherence** of beam
- optical resonator determines frequency of oscillation $\rightarrow$ **tunability**

Cavity Lifetime: Rate Equation of Empty Cavity

First we look at a light field created in an *empty* cavity, i.e. there is no light absorption or amplification.

The propagating light bounces around inside the cavity. Some of the light is lost through scattering, diffraction at the cavity edges, etc; also some is emitted through the partially transmitting mirrors. The light intensity in the cavity therefore gradually decays.

If the transmission of the mirrors is sufficiently small, the decay can be represented as a distributed spatial loss, arising from internal losses (α_i) and mirror losses (α_m):

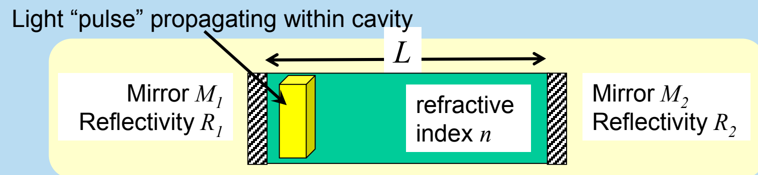
$$\frac{d\varphi(z)}{dz} = -(\alpha_m + \alpha_i)\varphi(z)$$

Equivalently, the loss can be represented by a finite cavity lifetime τ_{cav} in the time domain:

$$\frac{d\varphi(t)}{dt} = -\frac{\varphi(t)}{\tau_{cav}}$$

We can see how this works by considering a short “pulse” of light propagating in the laser cavity:

Calculation of Cavity Lifetime



Zero time: $t=0$ $\qquad \phi(z) = \phi_0$

Single pass: $t=(nL/c)$ $\qquad \phi(z+L) = R_2\phi_0$

Round trip: $t=(2nL/c)$ $\qquad \phi(z+2L) = R_1R_2\phi_0$

$$\tau_{cav} = -\frac{dt}{d\varphi}\varphi \approx -\varphi\left(\frac{\Delta t}{\Delta\varphi}\right)_{RT} \quad \text{if} \begin{cases}(\Delta t)_{RT} << \tau_{cav} \\ (\Delta\varphi/\varphi)_{RT} << 1 \text{ (i.e. } R_1R_2 \approx 1)\end{cases}$$

$$\therefore \tau_{cav} \approx -\left(\frac{\varphi}{\Delta\varphi}\right)_{RT}(\Delta t)_{RT} = -\frac{\varphi(z)}{\varphi(z+2L)-\varphi(z)}(2nL/c)$$

$$\tau_{cav} \approx \frac{(2nL/c)}{1-R_1R_2}$$

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Cavity Lifetime 2

Exercises:

1. For a symmetric high reflectivity cavity ($R_1 = R_2 = R \approx 1$), show that the cavity lifetime is given by

$$\tau_{cav} \approx \frac{(nL/c)}{1-R}$$

2. Internal losses also affect the cavity lifetime. By writing the lifetime in terms of the internal losses (α_i) and mirror (α_m) losses, show that

$$\tau_{cav} = \frac{(2nL/c)}{2\alpha_i L - \ln(R_1R_2)}$$

3. Show that the formula in (2) approximates to

$$\tau_{cav} \approx \frac{(2nL/c)}{1-R_1R_2}$$

when $R_1 R_2 \approx 1$ and α_i is small.

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Rate equation for laser material in a cavity

We combine the terms for the radiative transition rates and the cavity loss (spontaneous emission can be neglected since it is isotropic and so couples weakly to the laser cavity mode)

$$\frac{d\phi}{dt} = \left(\frac{d\phi}{dt}\right)_{\text{radiative}} + \left(\frac{d\phi}{dt}\right)_{\text{cavity loss}} = \left(N_2 - \frac{g_2}{g_1}N_1\right)B_{21}\cdot\rho - \frac{\phi}{\tau_{cav}}$$

Converting to irradiance:

$$\frac{dI}{dt} = h\upsilon\cdot\left(N_2 - \frac{g_2}{g_1}N_1\right)B_{21}\cdot I - \frac{I}{\tau_{cav}}$$

In terms of optical gain γ:

$$\gamma = \left(N_2 - \frac{g_2}{g_1}N_1\right)\frac{c^2 A_{21}}{8\pi\nu^2}$$

$$\frac{dI}{dt} = \frac{8\pi\nu^2 h\upsilon\cdot}{c^2}\frac{B_{21}}{A_{21}}\gamma I - \frac{I}{\tau_{cav}}$$

Using Einstein's relation for A_{21}/B_{21}:

$$\frac{dI}{dt} = c\gamma I - \frac{I}{\tau_{cav}}$$

… a rate equation for laser material in a cavity

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Lasing threshold and gain clamping

We could have obtained this result more directly if we converted to the ***spatial*** gain (dI/dz) and remembered the definition of the gain coefficient:

$$\frac{dI}{dt} = \left(\frac{dI}{dt}\right)_{\text{radiative}} + \left(\frac{dI}{dt}\right)_{\text{cavity loss}} = c\left(\frac{dI}{dz}\right)_{\text{radiative}} + \left(\frac{dI}{dt}\right)_{\text{cavity loss}} = c\gamma I - \frac{I}{\tau_{cav}}$$

In steady state ($dI/dt=0$), there are **two solutions** to the equation:

$$\left(c\gamma - \frac{1}{\tau_{cav}}\right)I = 0$$

$I=0$

There is **no light** (neglecting spontaneous emission) in the laser cavity, i.e. the optical loss exceeds the optical gain.

BELOW THRESHOLD FOR LASING

I is finite

There is light of intensity I inside the laser cavity, and the gain is **clamped** to a fixed value:

$$\gamma_{th} = \frac{1}{c\tau_{cav}}$$

AT OR ABOVE LASING THRESHOLD

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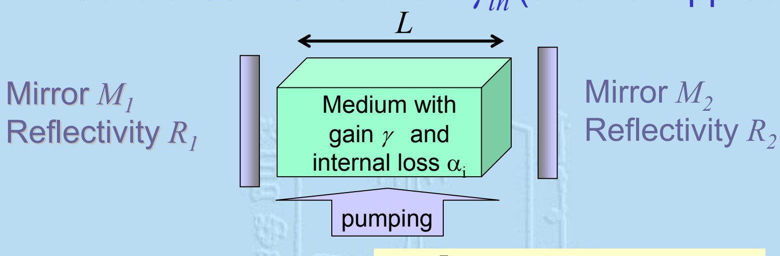
Threshold Gain Coefficient γ_{th} (another approach)



Round-trip gain G:

$$G = \frac{I_2}{I_1} = R_1 R_2 \exp[2(\gamma - \alpha_i)L]$$

Self-sustained oscillation when round trip gain $G = 1$ and $\gamma = \gamma_{th}$

$$R_1 R_2 \exp[2(\gamma_{th} - \alpha_i)L] = 1$$

$$\therefore \gamma_{th} = \alpha_i + \frac{1}{2L}\ln\left(\frac{1}{R_1 R_2}\right) = \alpha_i + \alpha_m$$

where α_m is the mirror loss:

$$\alpha_m = \frac{1}{2L}\ln\left(\frac{1}{R_1 R_2}\right) = -\frac{1}{2L}\ln(R_1 R_2) = \frac{1}{2L}\left|\ln(R_1 R_2)\right|$$

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Gain coefficient and light intensity

We earlier found the spatial gain coefficient:

$$\gamma = \left(N_2 - \frac{g_2}{g_1} N_1\right)\sigma_0$$

and the population inversion for a 4-level system:

$$N_2 - \frac{g_2}{g_1} N_1 = \frac{R}{1/\tau_{sp} + W}$$

Hence

$$\gamma = \left(\frac{R}{1/\tau_{sp} + W}\right)\sigma_0 = \frac{R\tau_{sp}\sigma_0}{1 + \dfrac{I(\nu)}{c/B_{21}\tau_{sp}}} = \frac{R\tau_{sp}\sigma_0}{1 + \dfrac{I(\nu)}{I_{sat}}}$$

Below lasing threshold

- light density in laser cavity is zero
- gain increases linearly with pumping:

$$\begin{cases} \gamma = R\tau_{sp}\sigma_0 = \dfrac{c^2}{8\pi\nu^2} R \\ I = 0 \end{cases}$$

Above lasing threshold:

- gain is clamped at γ_{th}
- light density increases with pumping rate for $R > R_{th}$:

$$\begin{cases} \gamma = \gamma_{th} \\ I = I_{sat}\left(\dfrac{R\tau_{sp}\sigma_0 - \gamma_{th}}{\gamma_{th}}\right) = I_{sat}\left(\dfrac{R}{R_{th}} - 1\right) \end{cases}$$

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Emitted laser power

Power emitted by stimulated emission in a lasing mode of volume V is:

$$P_e = V \cdot h\nu \cdot \left(\frac{d\phi}{dt}\right)_{stim} = V \cdot h\nu \cdot \left(N_2 - \frac{g_2}{g_1} N_1\right) \cdot W$$

Hence above threshold $(R>R_{th})$ in an ideal 4-level laser:

$$P_e = V \cdot h\nu \cdot \frac{\gamma_{th}}{\sigma_0} \cdot \left(R\frac{\sigma_0}{\gamma_{th}} - 1/\tau_{sp}\right) = V \cdot h\nu \cdot \left(R - \frac{\gamma_{th}}{\sigma_0 \tau_{sp}}\right) = V \cdot h\nu \cdot (R - R_{th})$$

The power from spontaneous emission at threshold is similarly:

$$P_s = V \cdot h\nu \cdot \left(\frac{d\phi}{dt}\right)_{spon} = V \cdot h\nu \cdot \frac{N_2}{\tau_{sp}} \approx V \cdot h\nu \cdot \frac{\gamma_{th}}{\sigma_0 \tau_{sp}} = V \cdot h\nu \cdot R_{th}$$

Hence a very simple expression for the power emitted from an ideal 4-level laser:

$$P_e = P_s \cdot \left(\frac{R}{R_{th}} - 1\right) \qquad (R>R_{th})$$

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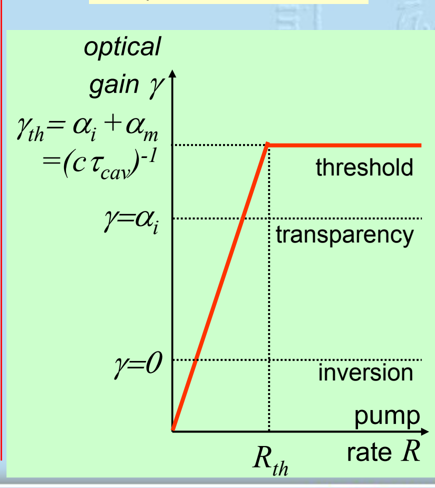
Optical gain and emitted power of a laser

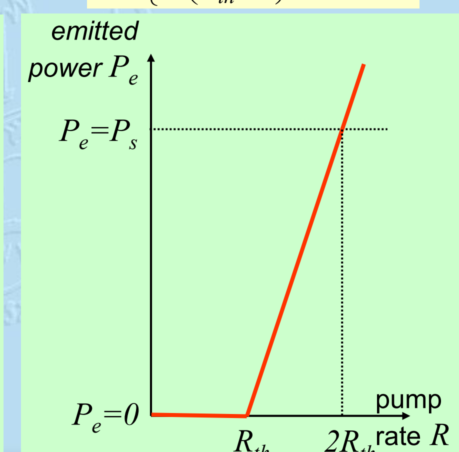
gain:

$$\gamma = \begin{cases} R\tau_{21}\sigma_0 & (R < R_{th}) \\ \gamma_{th} & (R \geq R_{th}) \end{cases}$$

power:

$$P_e = \begin{cases} 0 & (R < R_{th}) \\ P_s\left(\frac{R}{R_{th}} - 1\right) & (R \geq R_{th}) \end{cases}$$





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Output power: dependence on mirror reflectivities

We've focussed on how the laser output power depends on the pumping rate R, but we can also work out how it depends on the mirror reflectivities. And for a given laser material or configuration, we can work out how reflective to make the mirrors, in order to optimise the light output!

We know the following:

1. The gain saturates with power according to the equation: where P_S is the power associated with spontaneous emission at threshold. $$\gamma = \frac{\gamma_0}{1 + P_e / P_s}$$
2. P_S scales with the threshold gain γ_{th} : $$P_s \approx V \cdot h\nu \cdot \gamma_{th} / \sigma_0 \tau_{sp} = P'_s \gamma_{th}$$
3. At threshold, the gain balances the losses: $$\gamma_{th} = \alpha_m + \alpha_i$$

Rearranging (1) and taking account of (2), we obtain:

$$P_e = P_s \left(\frac{\gamma_0}{\gamma_{th}} - 1 \right) = P'_s \gamma_{th} \left(\frac{\gamma_0}{\gamma_{th}} - 1 \right)$$

Substituting (3) we get:

$$P_e = P'_s (\alpha_m + \alpha_i) \left(\frac{\gamma_0}{(\alpha_m + \alpha_i)} - 1 \right)$$

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Output power (2)

We have *calculated* P_e, the power emitted by stimulated emission within the laser.

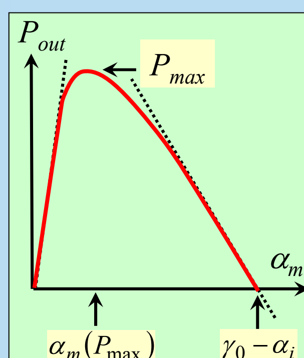
What we can *measure* is P_{out}, the light coming out through the mirrors. This is just given by the fraction of the total light which couples into mirror loss rather than internal loss:

$$P_{out} = \left(\frac{\alpha_m}{\alpha_m + \alpha_i} \right) P_e = P'_s \alpha_m \left(\frac{\gamma_0}{(\alpha_m + \alpha_i)} - 1 \right)$$

How does this vary with the mirror loss α_m ?
Look at the limiting cases:

- When α_m is zero: no light gets out of the cavity!
- When α_m is small: P_{out} *increases* linearly with α_m $$P_{out} \approx P'_s \alpha_m (\gamma_0 / \alpha_i - 1)$$
- When α_m is large: P_{out} *decreases* linearly with α_m (think about *why*.. $$P_{out} \approx P'_s (\gamma_0 - \alpha_m)$$

Hence there must be a value of α_m which maximises the output power



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Output power (3): (important) exercise

A simple laser contains a medium with a gain coefficient (at low optical power) of 0.1cm^{-1}, and an internal loss coefficient of $\alpha_i = 0.02\ \text{cm}^{-1}$. The gain medium fills the space between two parallel plane mirrors, one with reflection coefficient of R_1=100% and the other with a reflection coefficient R_2.

- Find an expression for the mirror loss α_m which maximises the output power.
- Calculate the corresponding mirror reflectivity R_2.
- Calculate the corresponding threshold gain γ_{th} .
- Find an expression for the maximum output power P_{max} .

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What U need 2 know: LASER oscillation

Describe briefly: ***Role of cavity in a laser***

- **multiple passes** for sufficient amplification
- **optical feedback** gives self-sustained oscillation
- resonant cavity $\rightarrow$ **tunability**

Definition of: ***cavity losses***

$$\frac{d\varphi(t)}{dt} = -\frac{\varphi(t)}{\tau_{cav}}$$

Derivation of: ***cavity lifetime***

$$\tau_{cav} \approx \frac{(2nL/c_0)}{1-R_1R_2} \approx \frac{(nL/c_0)}{1-R}$$

Derive: ***rate equation for optical gain medium in cavity***

$$\frac{d\varphi(t)}{dt} = \left[\left(\frac{c_0}{n}\right)\gamma - \frac{1}{\tau_{cav}}\right]\varphi(t)$$

Derive: ***steady state solution***

Solution 1: $\phi=0$ (below threshold)

Solution 2: $\phi>0$ (gain is pinned at threshold value)

$$\gamma_{th} = n/c_0\tau_{cav}$$

Derivation of: ***optical gain at threshold*** (in terms of mirror losses)

$$\gamma_{th} = \alpha_i + \frac{1}{2L}\ln\left(\frac{1}{R_1R_2}\right) = \alpha_i + \alpha_m$$

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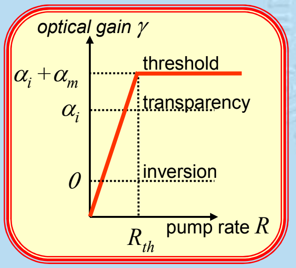
What U need 2 know: LASER oscillation

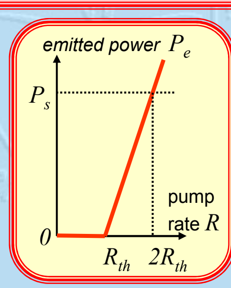
Derive:
Dependence of optical gain on pump rate

$$\gamma = \begin{cases} R\tau_{21}\sigma_0 & (R < R_{th}) \\ \gamma_{th} & (R \geq R_{th}) \end{cases}$$



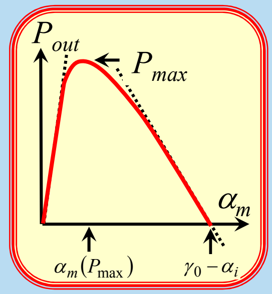
Derive:
Dependence of emitted power on pump rate

$$P_e = \begin{cases} 0 & (R < R_{th}) \\ P_s\left(\dfrac{R}{R_{th}} - 1\right) & (R \geq R_{th}) \end{cases}$$



Derive:
Dependence of laser output power on mirror losses

$$P_{out} \propto P_s' \alpha_m \left(\frac{\gamma_0}{(\alpha_m + \alpha_i)} - 1 \right)$$



What U need 2 know: LASER oscillation

Describe briefly:
Role of cavity in a laser

- **multiple passes** for sufficient amplification
- **optical feedback** gives self-sustained oscillation
- resonant cavity $\rightarrow$ **tunability**

Definitions of:

cavity losses

$$\frac{d\varphi(z)}{dz} = -(\alpha_m + \alpha_i)\varphi(z) \quad \text{(in spatial domain)}$$

cavity lifetime

$$\frac{d\varphi(t)}{dt} = -\frac{\varphi(t)}{\tau_{cav}} \quad \text{(in time domain)}$$

Derivation of:
cavity lifetime

$$\tau_{cav} \approx \frac{(2nL/c_0)}{1 - R_1 R_2} \approx \frac{(nL/c_0)}{1 - R}$$

(approximate forms)

What U need 2 know: LASER oscillation

Derive:
rate equation for optical gain medium in cavity

$$\begin{cases} \dfrac{d\varphi(z)}{dz} = [\gamma - (\alpha_m + \alpha_i)]\varphi(z) & \text{(spatial domain)} \\ \dfrac{d\varphi(t)}{dt} = \left[\left(\dfrac{c_0}{n}\right)\gamma - \dfrac{1}{\tau_{cav}}\right]\varphi(t) & \text{(time domain)} \end{cases}$$

optical gain *cavity loss*

Derive:
steady state solution

Solution 1: $\phi = 0$ (below threshold)

Solution 2: $\phi > 0$ (gain is pinned at threshold value)

$$\begin{cases} \gamma_{th} = \alpha_m + \alpha_i \\ \gamma_{th} = n/c_0\tau_{cav} \end{cases}$$

Derivation of:
optical gain at threshold
(in terms of mirror losses)

$$\therefore \gamma_{th} = \alpha_i + \frac{1}{2L}\ln\left(\frac{1}{R_1 R_2}\right) = \alpha_i + \alpha_m$$

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