Chapter 4

Continuous-Time Fourier Transform

Introduction

- *Periodic* signals are represented as linear combination of harmonically related complex exponentials (*Fourier Series*).
- Non-periodic (*Aperiodic*) signals are represented by complex exponentials using *Fourier Transform*.

Fourier Transform of $x(t)$ → $$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$$

Inverse Fourier Transform of $X(j\omega)$ → $$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\,d\omega$$

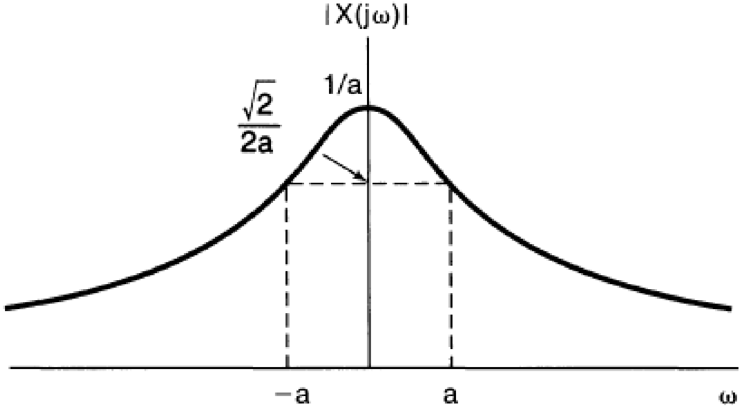
Fourier Transform - Example 1

Consider the signal $x(t) = e^{-at}u(t), \ \ a > 0$

Find its Fourier transform.

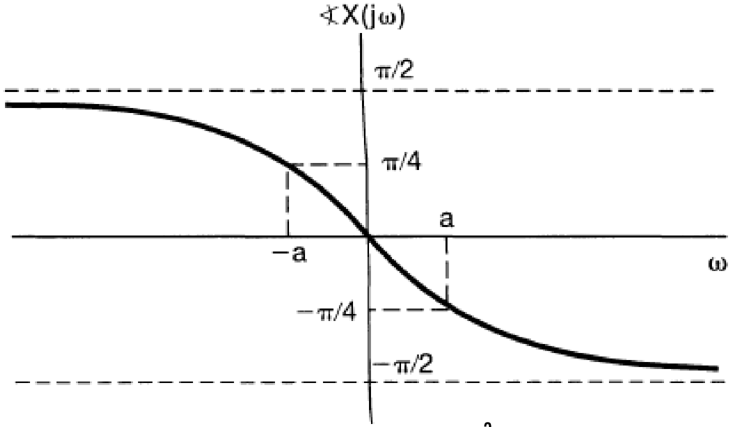
Fourier transform $X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$

$$= \int_{-\infty}^{\infty} e^{-at} u(t) e^{-j\omega t} dt = \int_{0}^{\infty} e^{-(a+j\omega)t} dt = \left. \frac{e^{-(a+j\omega)t}}{-(a+j\omega)} \right|_0^{\infty} = \frac{1}{a+j\omega}$$



Module $|X(j\omega)| = \left| \frac{1}{a+j\omega} \right| = \left| \frac{a-j\omega}{a^2+\omega^2} \right| = \sqrt{\frac{a^2}{(a^2+\omega^2)^2} + \frac{\omega^2}{(a^2+\omega^2)^2}} = \frac{1}{\sqrt{a^2+\omega^2}}$

Phase $\angle X(j\omega) = \tan^{-1}\left(\frac{-\omega}{a^2+\omega^2} / \frac{a}{a^2+\omega^2} \right) = -\tan^{-1}\left(\frac{\omega}{a}\right)$

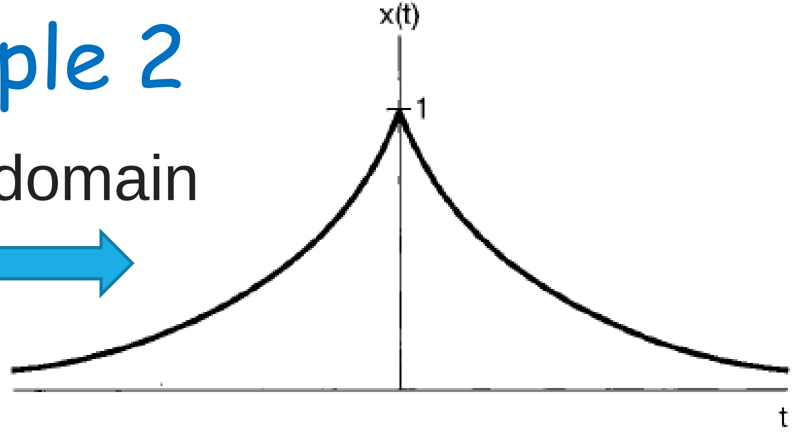


Fourier Transform - Example 2

Find Fourier transform of:

$$x(t) = e^{-a|t|}\ ,\ a > 0$$

Time-domain



The Fourier transform of the signal is:

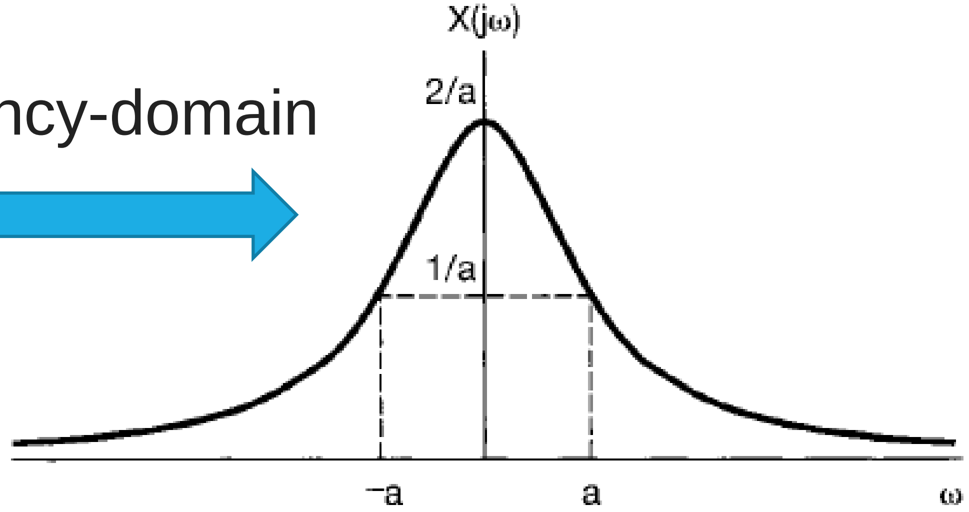
$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} e^{-a|t|} e^{-j\omega t} dt$$

$$= \int_{-\infty}^{0} e^{at} e^{-j\omega t} dt + \int_{0}^{\infty} e^{-at} e^{-j\omega t} dt$$

$$= \left. \frac{e^{(a-j\omega)t}}{(a-j\omega)} \right|_{-\infty}^{0} + \left. \frac{e^{-(a+j\omega)t}}{-(a+j\omega)} \right|_{0}^{\infty}$$

$$= \frac{1}{a-j\omega} + \frac{1}{a+j\omega} = \frac{a+j\omega+a-j\omega}{a^2+\omega^2} \qquad = \frac{2a}{a^2+\omega^2}$$

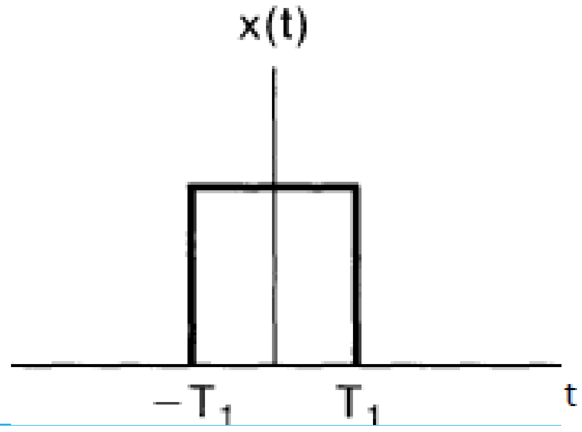
Frequency-domain



Fourier Transform - Example 3

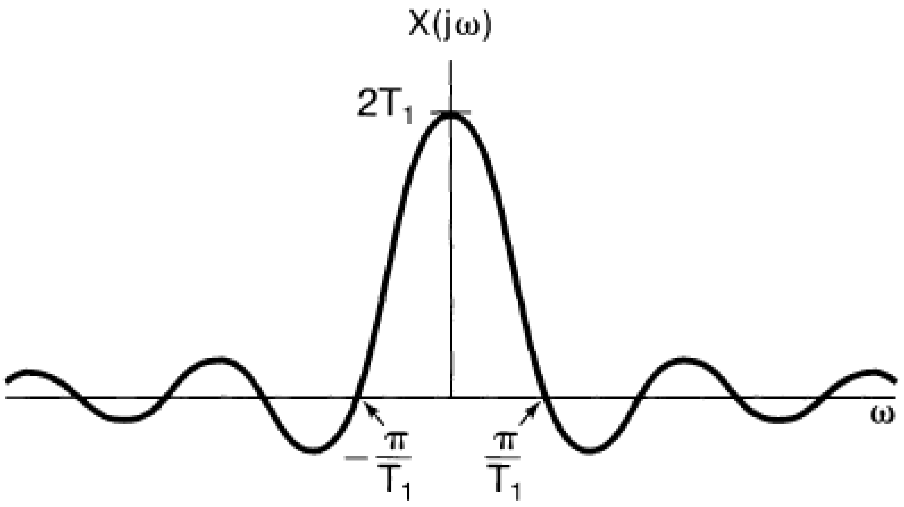
Find Fourier transform of:

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & |t| > T_1 \end{cases}$$



$$X(j\omega) = \int_{-T_1}^{T_1} 1.e^{-j\omega t}dt = \left.\frac{e^{-j\omega t}}{-j\omega}\right|_{-T_1}^{T_1} = -\frac{1}{j\omega}\left[e^{-j\omega T_1} - e^{j\omega T_1}\right]$$

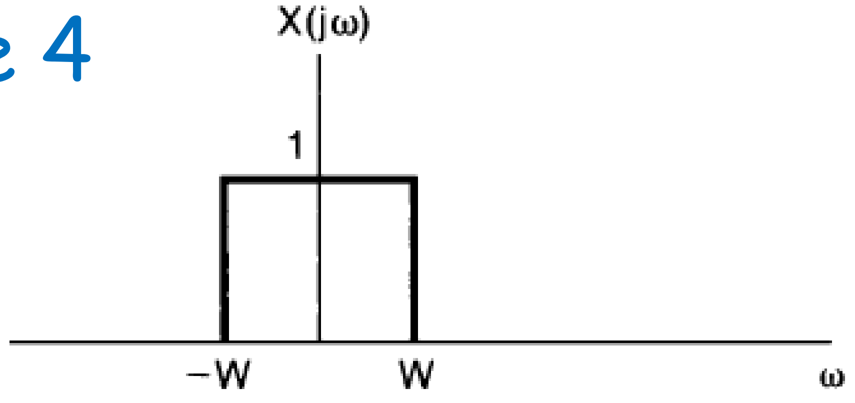
$$= \frac{1}{j\omega}\left[e^{j\omega T_1} - e^{-j\omega T_1}\right] = \frac{2j\sin(\omega T_1)}{j\omega} = \frac{2\sin(\omega T_1)}{\omega}$$



Fourier Transform - Example 4

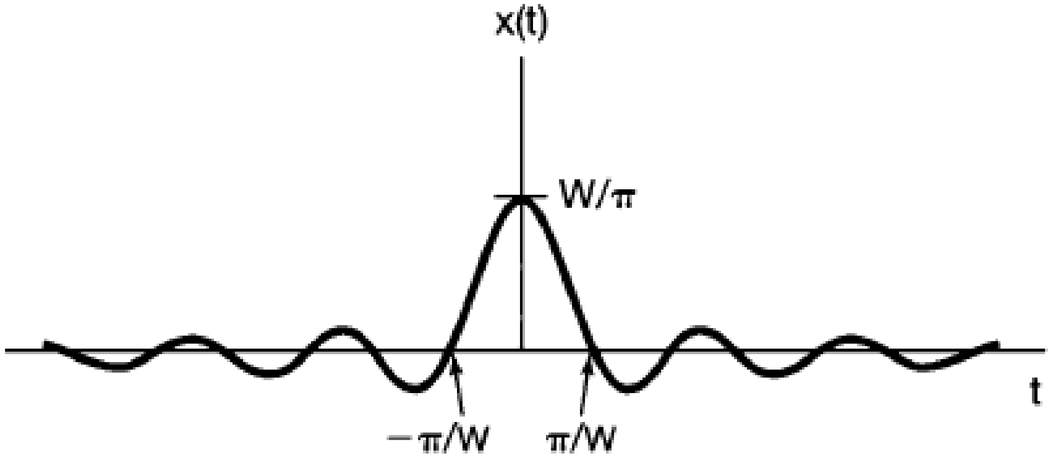
Find the inverse Fourier transform of:

$$X(j\omega) = \begin{cases} 1, & |\omega| < W \\ 0, & |\omega| > W \end{cases}$$



the inverse Fourier transform

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega$$



$$x(t) = \frac{1}{2\pi}\int_{-W}^{W} e^{j\omega t}\, d\omega = \frac{1}{2\pi}\left[\frac{e^{j\omega t}}{jt}\right]_{-W}^{W}$$

$$x(t) = \frac{1}{2\pi}\left[\frac{e^{jWt}}{jt} - \frac{e^{-jWt}}{-jt}\right] = \frac{1}{\pi t}\left[\frac{e^{jWt} - e^{-jWt}}{j2}\right] = \frac{W}{\pi}\frac{sin(Wt)}{Wt}$$

Problem 1

Use the Fourier transform analysis equation to calculate the Fourier transforms of :

$$\text{(a)}\ \ e^{-2(t-1)}u(t-1) \qquad\qquad \text{(b)}\ \ e^{-2|t-1|}$$

(a) $\quad e^{-2(t-1)}u(t-1)$

$$X(j\omega)=\int_{-\infty}^{\infty}x(t)e^{-j\omega t}dt=\int_{-\infty}^{\infty}e^{-2(t-1)}u(t-1)e^{-j\omega t}dt$$

$\tau = t-1$

$$=\int_{-\infty}^{\infty}e^{-2\tau}u(\tau)e^{-j\omega(\tau+1)}d\tau=e^{-j\omega}\int_{-\infty}^{\infty}e^{-2\tau}u(\tau)e^{-j\omega\tau}d\tau$$

$$=e^{-j\omega}\int_{0}^{\infty}e^{-(2+j\omega)\tau}d\tau=e^{-j\omega}\left.\frac{e^{-(2+j\omega)\tau}}{-(2+j\omega)}\right|_{0}^{\infty}$$

$$=\frac{e^{-j\omega}}{2+j\omega}$$

(b) $\quad e^{-2|t-1|}$

$$X(j\omega)=\int_{-\infty}^{\infty}x(t)e^{-j\omega t}dt=\int_{-\infty}^{\infty}e^{-2|t-1|}e^{-j\omega t}dt$$

$$=\int_{-\infty}^{\infty}e^{-2|\tau|}e^{-j\omega(\tau+1)}d\tau=e^{-j\omega}\int_{-\infty}^{\infty}e^{-2|\tau|}e^{-j\omega\tau}d\tau$$

$$=e^{-j\omega}\int_{-\infty}^{0}e^{2\tau}.e^{-j\omega\tau}d\tau+e^{-j\omega}\int_{0}^{\infty}e^{-2\tau}.e^{-j\omega\tau}d\tau$$

$$=e^{-j\omega}\left[\left.\frac{e^{(2-j\omega)\tau}}{(2-j\omega)}\right|_{-\infty}^{0}+\left.\frac{e^{-(2+j\omega)\tau}}{-(2+j\omega)}\right|_{0}^{\infty}\right]$$

$$=\frac{4e^{-j\omega}}{4+\omega^2}$$

Problem 2

Use the Fourier transform analysis equation to calculate the Fourier transforms of :

$$\delta(t+1)+\delta(t-1)$$

$$X(j\omega)=\int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

$$=\int_{-\infty}^{\infty}[\delta(t+1)+\delta(t-1)]e^{-j\omega t}dt$$

$$=\int_{-\infty}^{\infty}\delta(t+1)e^{-j\omega t}dt+\int_{-\infty}^{\infty}\delta(t-1)]e^{-j\omega t}dt$$

$$=e^{-j\omega(-1)}+e^{-j\omega(1)}=e^{j\omega}+e^{-j\omega}$$

$$=2\times\left(\frac{e^{j\omega}+e^{-j\omega}}{2}\right)=2\cos\omega$$

$$|X(j\omega)|=2\,|\cos(\omega)|$$

Some useful Fourier transform:

$$\boxed{\begin{aligned}&x(t)=\delta(t)\\&\Rightarrow X(j\omega)=1\end{aligned}}$$

$$x(t)=\delta(t+1)$$

$$\Rightarrow X(j\omega)=e^{j\omega}$$

$$x(t)=\delta(t-1)$$

$$\Rightarrow X(j\omega)=e^{-j\omega}$$

Problem 3

Use the Fourier transform analysis equation to calculate the Fourier transforms of :

$$\frac{d}{dt}\{u(-2-t)+u(t-2)\}$$

$$X(j\omega)=\int_{-\infty}^{\infty}\frac{d}{dt}\{u(-2-t)+u(t-2)\}e^{-j\omega t}dt$$

$$=\int_{-\infty}^{\infty}-\delta(-2-t)e^{-j\omega t}dt+\int_{-\infty}^{\infty}\delta(t-2)e^{-j\omega t}dt$$

$$=I_1+I_2$$

$$I_1=\int_{-\infty}^{\infty}-\delta(-2-t)e^{-j\omega t}dt,\ \text{Let } \tau=-2-t, d\tau=-dt$$

$$=\int_{\infty}^{-\infty}-\delta(\tau)e^{j\omega(\tau+2)}(-d\tau)=-e^{2j\omega}\int_{-\infty}^{\infty}\delta(\tau)e^{j\omega\tau}d\tau=-e^{2j\omega}$$

$$I_2=\int_{-\infty}^{\infty}\delta(t-2)e^{-j\omega t}dt=e^{-2j\omega}$$

$$X(j\omega)=-e^{2j\omega}+e^{-2j\omega}$$

$$=-2j\times\left(\frac{e^{2j\omega}-e^{-2j\omega}}{2j}\right)$$

$$=-2j\sin(2\omega)$$

$$|X(j\omega)|=2|\sin(2\omega)|$$

Fourier Transform for Periodic Signals

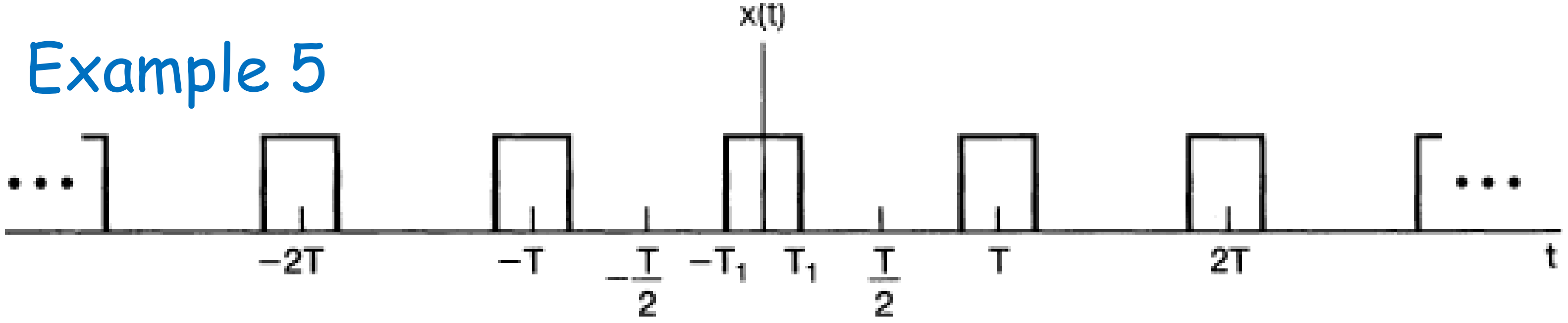
Obtain the Fourier transform of a periodic signal $x(t)$ directly from its Fourier series a_k

$$X(j\omega)=\sum_{k=-\infty}^{\infty} a_k\, 2\pi\delta(\omega-k\omega_0) \xrightarrow{\text{Inverse FT}} x(t)=\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

Proof

$$x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\sum_{k=-\infty}^{\infty} a_k 2\pi\delta(\omega-k\omega_0)\, e^{j\omega t}d\omega=\sum_{k=-\infty}^{\infty} a_k\frac{2\pi}{2\pi}\int_{-\infty}^{\infty}\delta(\omega-k\omega_0)e^{j\omega t}d\omega=\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

Example 5

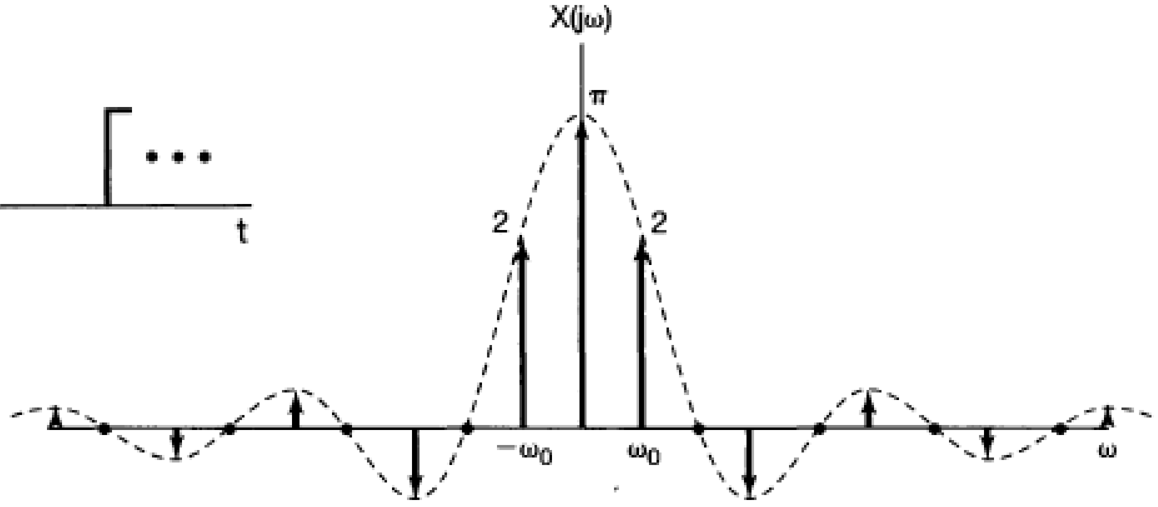


The Fourier series coefficients of the above periodic signal:

$$a_k=\frac{\sin(k\omega_0 T_1)}{\pi k}$$

Fourier transform

$$X(j\omega)=\sum_{k=-\infty}^{\infty} 2\pi\frac{\sin(k\omega_0 T_1)}{\pi k}\delta(\omega-k\omega_0)=\sum_{k=-\infty}^{\infty} 2\frac{\sin(k\omega_0 T_1)}{k}\delta(\omega-k\omega_0)$$



Fourier transform of a symmetric periodic square wave

Example 6

Obtain the Fourier transform of $x(t) = \sin(\omega_0 t)$

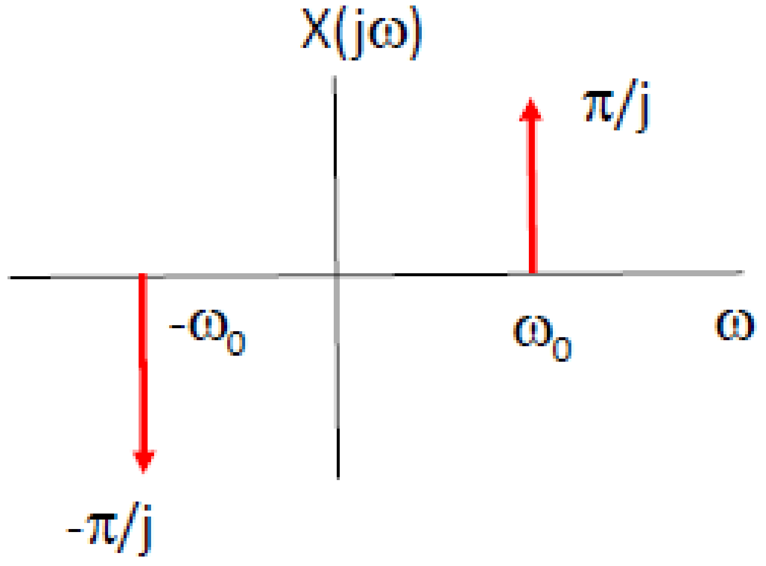
$$x(t) = \sin(\omega_0 t) = \frac{1}{2j} e^{j\omega_0 t} - \frac{1}{2j} e^{-j\omega_0 t}$$

Fourier series coefficients: $a_1 = \frac{1}{2j}, a_{-1} = -\frac{1}{2j}, a_k = 0 \text{ for } |k| \neq 1$

Fourier transform $X(j\omega) = \sum_{k=-\infty}^{\infty} a_k 2\pi\delta(\omega - k\omega_0)$

$$= \frac{2\pi}{2j}\delta(\omega - \omega_0) - \frac{2\pi}{2j}\delta(\omega + \omega_0)$$

$$= \frac{\pi}{j}\delta(\omega - \omega_0) - \frac{\pi}{j}\delta(\omega + \omega_0)$$



Example 6

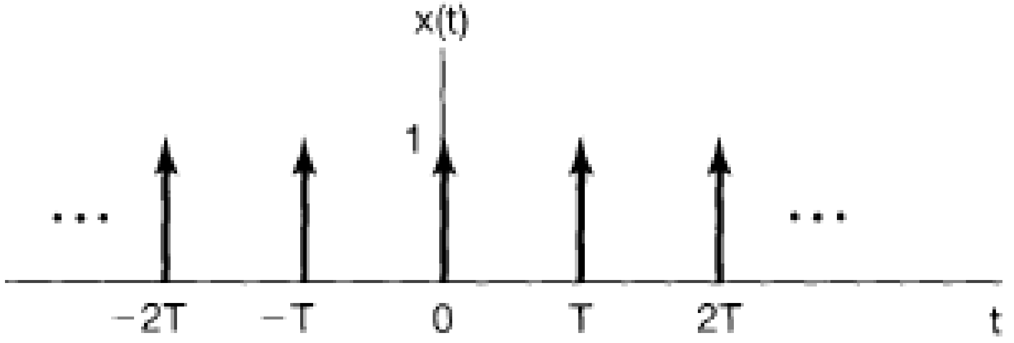
Obtain the Fourier transform of the impulse train

$$x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$$

Solution:

The Fourier series coefficients for this signal are:

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t)e^{-jk\omega_0 t}dt = \frac{1}{T}e^0 = \frac{1}{T}$$

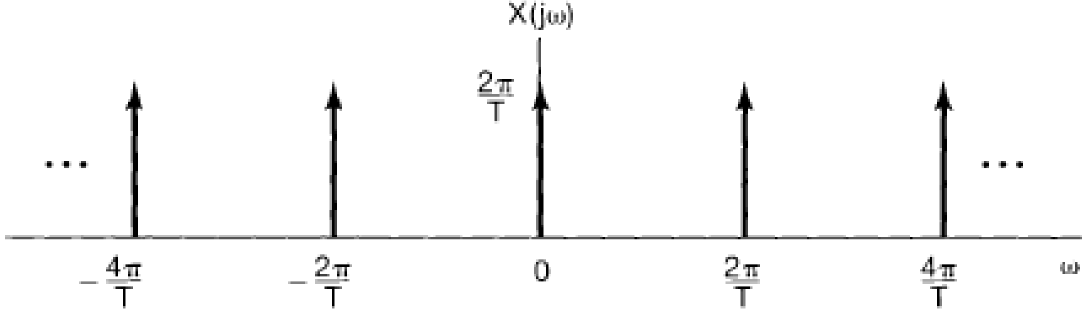


Impulse train with a period of T.

Therefore, the Fourier transform is:

$$X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0) = \frac{2\pi}{T}\sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_0)$$

$$= \frac{2\pi}{T}\sum_{k=-\infty}^{\infty} \delta(\omega - \frac{2\pi}{T}k)$$



Fourier transform of Periodic impulse train

Problem 4

Determine the Fourier transform of each of the following periodic signals.

(a) $\sin\left(2\pi t+\frac{\pi}{4}\right)$ (b) $1+\cos\left(6\pi t+\frac{\pi}{8}\right)$

Solution:

(a) $x(t)=\sin\left(2\pi t+\frac{\pi}{4}\right);$

comparing with

$$x(t)=\sin\left(\omega_0 t+\frac{\pi}{4}\right),$$

and $T=\frac{2\pi}{\omega_0}$ $\Rightarrow$ $\omega_0=2\pi; T=1$

$$x(t)=\frac{e^{j(2\pi t+\frac{\pi}{4})}-e^{-j(2\pi t+\frac{\pi}{4})}}{2j}$$

$$x(t)=\underbrace{\boxed{\left[\frac{e^{j\frac{\pi}{4}}}{2j}\right]}}_{a_1}e^{j2\pi t}+\underbrace{\boxed{\left[-\frac{e^{-j\frac{\pi}{4}}}{2j}\right]}}_{a_{-1}}e^{-j2\pi t}$$

$a_k=0$
all other k

The Fourier transform of the periodic signal

$$X(j\omega)=\sum_{k=-\infty}^{\infty}a_k 2\pi\delta(\omega-k\omega_0)$$

$$=\frac{\pi}{j}e^{j\frac{\pi}{4}}\delta(\omega-2\pi)-\frac{\pi}{j}e^{-j\frac{\pi}{4}}\delta(\omega+2\pi)$$

(b) $x(t)=1+\cos\left(6\pi t+\frac{\pi}{8}\right)$ $\Rightarrow$ $\omega_0=6\pi$

$$=e^{0}+\frac{e^{j(6\pi t+\pi/8)}+e^{-j(6\pi t+\pi/8)}}{2}$$

$$x(t)=\underbrace{\boxed{1}}_{a_0}e^{j0t}+\underbrace{\boxed{\frac{1}{2}e^{j\frac{\pi}{8}}}}_{a_1}e^{j6\pi t}+\underbrace{\boxed{\frac{1}{2}e^{-j\frac{\pi}{8}}}}_{a_{-1}}e^{-j6\pi t}$$

$a_k=0$
all other k

The Fourier transform of the periodic signal

$$X(j\omega)=\sum_{k=-\infty}^{\infty}a_k 2\pi\delta(\omega-k\omega_0)$$

$$=2\pi\delta(\omega)+\pi e^{j\frac{\pi}{8}}\delta(\omega-6\omega_0)+\pi e^{-j\frac{\pi}{8}}\delta(\omega+6\omega_0)$$

Problem 5

Use the Fourier synthesis equations to determine the Inverse Fourier transform of

$$X_1(j\omega) = 2\pi\delta(\omega) + \pi\delta(\omega - 4\pi) + \pi\delta(\omega + 4\pi)$$

(b) $$X_2(j\omega) = \begin{cases} 2, & 0 \le \omega < 2 \\ -2, & -2 \le \omega < 0 \\ 0, & |\omega| > 0 \end{cases}$$

Solution:

(a)

$$x_1(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}[2\pi\delta(\omega) + \pi\delta(\omega - 4\pi) + \pi\delta(\omega + 4\pi)]e^{j\omega t}d\omega$$

$$= \int_{-\infty}^{\infty}\delta(\omega)e^{j\omega t}d\omega + \frac{1}{2}\int_{-\infty}^{\infty}\delta(\omega - 4\pi)e^{j\omega t}d\omega$$

$$+ \frac{1}{2}\int_{-\infty}^{\infty}\delta(\omega + 4\pi)e^{j\omega t}d\omega$$

$$= e^0 + \frac{1}{2}e^{j4\pi t} + \frac{1}{2}e^{-j4\pi t}$$

$$= 1 + \cos(4\pi t)$$

(b)

$$x_2(t) = \frac{1}{2\pi}\left[\int_0^2 2e^{j\omega t}d\omega + \int_{-2}^{0} -2e^{j\omega t}d\omega\right]$$

$$= \frac{1}{\pi}\left[\left.\frac{e^{j\omega t}}{jt}\right|_0^2 - \left.\frac{e^{j\omega t}}{jt}\right|_{-2}^0\right] = \frac{1}{\pi jt}\left[(e^{j2t} - 1) - (1 - e^{-j2t})\right]$$

$$= \frac{1}{\pi jt}\left[\left(e^{jt}\right)^2 + \left(e^{-jt}\right)^2 - 2e^{jt}e^{-jt}\right] = \frac{1}{\pi jt}\left(e^{jt} - e^{-jt}\right)^2$$

$$= \frac{-4}{\pi jt}\left(\frac{e^{jt} - e^{-jt}}{2j}\right) = \frac{-4}{j\pi t}\sin^2(t)$$

Properties of CT Fourier Transform

- Give insight of the relationship between the time-domain and frequency-domain descriptions of a signal.
- Are useful in reducing the complexity of the evaluation of Fourier transforms or inverse transforms.

1. Linearity

If $x(t) \overset{\mathcal{F}}{\Leftrightarrow} X(j\omega)$, $y(t) \overset{\mathcal{F}}{\Leftrightarrow} Y(j\omega)$ Than $a\,x(t) + b\,y(t) \overset{\mathcal{F}}{\Leftrightarrow} a\,X(j\omega) + b\,Y(j\omega)$

2. Time shifting

$$x(t - t_0) \overset{\mathcal{F}}{\Leftrightarrow} e^{-j\omega t_0}\, X(j\omega)$$

Proof $\Rightarrow$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

$$\Rightarrow x(t - t_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega(t - t_0)}d\omega$$

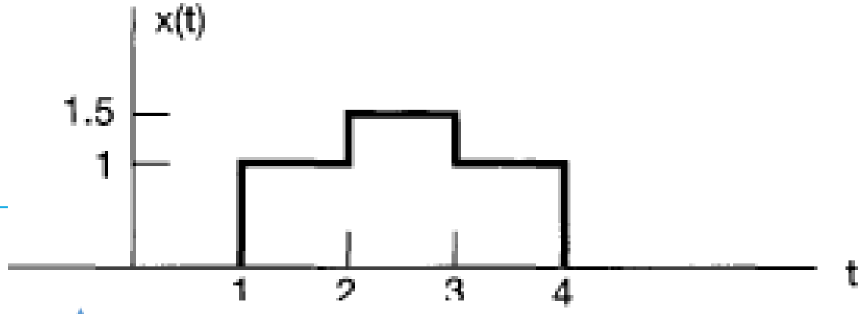
$$\Rightarrow x(t - t_0) = e^{-j\omega t_0}\frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

$$\Rightarrow \Im(x(t - t_0)) = e^{-j\omega t_0}X(j\omega)$$

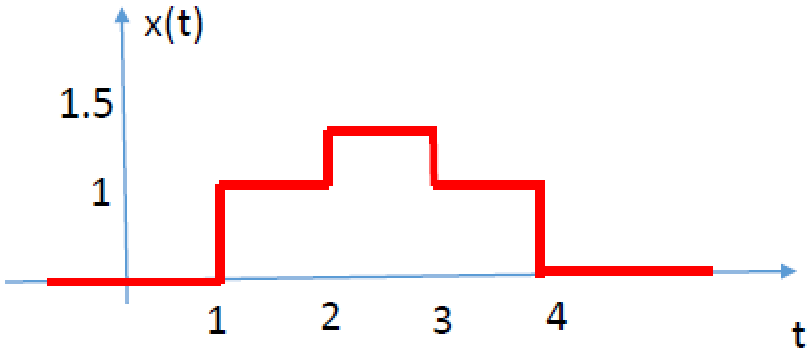
If a signal is time-shifted, the magnitude of the Fourier transform does not change; only there is a phase-shift in the Fourier transform.

Example 6

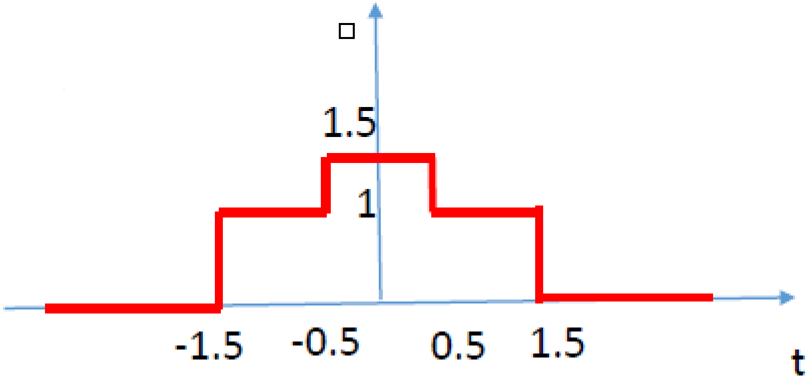
Evaluate the Fourier transform of $x(t)$



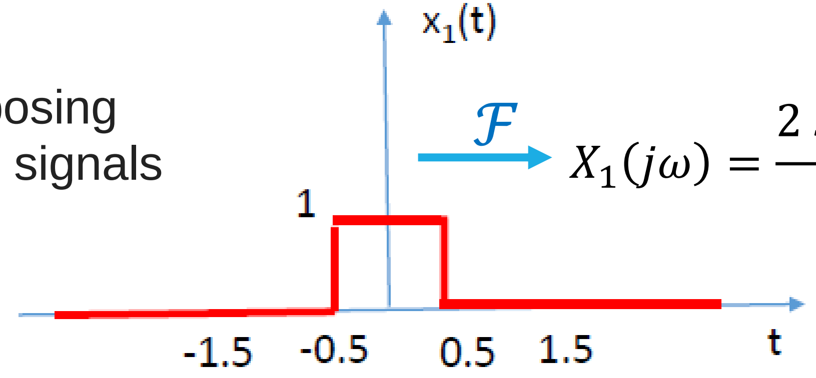
Solution:



By shifting 2.5

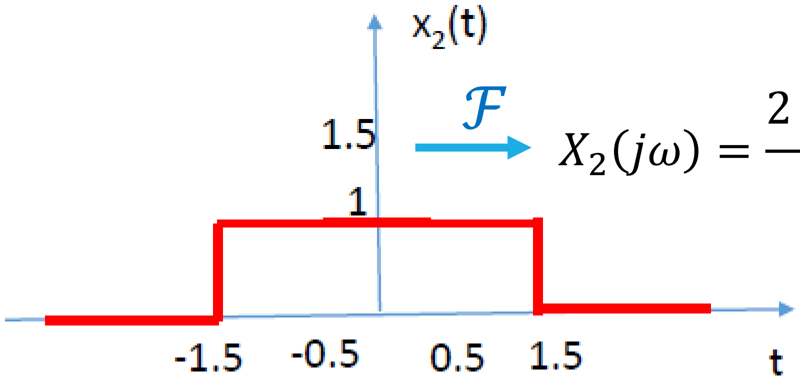


By Decomposing
into simpler signals



$$\xrightarrow{\mathcal{F}} X_1(j\omega) = \frac{2\,sin\left(\omega/_2\right)}{\omega}$$

$+$



$$\xrightarrow{\mathcal{F}} X_2(j\omega) = \frac{2\,sin\left(3\,\omega/_2\right)}{\omega}$$

$$x(t) = ½\, x_1(t - 2.5) + x_2(t - 2.5)$$

Example 6 - contd.

From Example 3 and the signals of the previous slide, we get:

$$X_1(j\omega) = \frac{2\,sin\left(\omega/_2\right)}{\omega} \quad \text{and} \quad X_2(j\omega) = \frac{2\,sin\left(3\,\omega/_2\right)}{\omega}$$

By linearity and time shifting properties:

$$X(j\omega) = \frac{1}{2}\mathcal{F}\{x_1(t-2.5)\} + \mathcal{F}\{x_2(t-2.5)\}$$

$$= \frac{1}{2}e^{-2.5j\omega}\mathcal{F}\{x_1(t)\} + e^{-2.5j\omega}\mathcal{F}\{x_2(t)\}$$

$$= \frac{1}{2}e^{-2.5j\omega}\,\frac{2\,sin\left(\omega/_2\right)}{\omega} + e^{-2.5j\omega}\,\frac{2\,sin\left(3\,\omega/_2\right)}{\omega}$$

$$= e^{-2.5j\omega}\left(\frac{sin\left(\omega/_2\right) + 2\,sin\left(3\,\omega/_2\right)}{\omega}\right)$$

Properties of CT Fourier Transform

3. Conjugation and Conjugate Symmetry

If $x(t) \overset{\mathcal{F}}{\Leftrightarrow} X(j\omega)$ Than $x^*(t) \overset{\mathcal{F}}{\Leftrightarrow} X^*(-j\omega)$

Proof

$$X^*(j\omega) = \left[\int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt\right]^*$$

$$= \int_{-\infty}^{\infty} x^*(t)e^{j\omega t}dt$$

Replace ω by -ω

$$X^*(-j\omega) = \int_{-\infty}^{\infty} x^*(t)e^{-j\omega t}dt$$

If $x(t)$ is real, $x(t) = x^*(t)$, then $X(j\omega)$ has conjugate symmetry: $X(-j\omega) = X^*(j\omega)$

Rectangular form of $X(j\omega)$ — *Polar form of $X(j\omega)$*

For real $x(t)$:

Real part of $X(j\omega)$

$\mathrm{Re}\{X(j\omega)\} = \mathrm{Re}\{X(-j\omega)\}$ ⇒ Even function of ω ⇐ $|X(j\omega)| = |X(-j\omega)|$

Imaginary part of $X(j\omega)$

$\mathrm{Im}\{X(j\omega)\} = -\mathrm{Im}\{X(-j\omega)\}$ ⇒ Odd function of ω ⇐ $\angle X(j\omega) = -\angle X(-j\omega)$

From positive frequencies we can determine magnitude and phase of $X(j\omega)$ for negative frequencies

For real and even $x(t)$: $x(t) \overset{\mathcal{F}}{\Leftrightarrow} X(j\omega)$

$X^*(j\omega) = X(j\omega)$ — real

$X(-j\omega) = X(j\omega)$ — even

For $x(t)$ real and odd

$X(j\omega)$ is purely imaginary and odd

For $x(t) = x_e(t) + x_o(t)$

Even $\{x(t)\} \overset{\mathcal{F}}{\Leftrightarrow}$ Re $\{X(j\omega)\}$

Odd $\{x(t)\} \overset{\mathcal{F}}{\Leftrightarrow}$ j $\mathrm{Im}\{X(j\omega)\}$

Example 7

Evaluate the Fourier transform of $x(t) = e^{-a|t|}$ for $a > 0$

From Example 6: $x(t) = e^{-a|t|}$ for $a > 0$ $\xrightarrow{\text{we have}}$ $x_1(t) = e^{-at}u(t) \overset{\mathcal{F}}{\Leftrightarrow} \dfrac{1}{a + j\omega}$

$$x(t) = e^{-a|t|} = \underset{t>0}{e^{-at}u(t)} + \underset{t<0}{e^{at}u(-t)} = 2\left[\frac{e^{-at}u(t) + e^{at}u(-t)}{2}\right] = 2\,Even\{e^{-at}u(t)\}$$

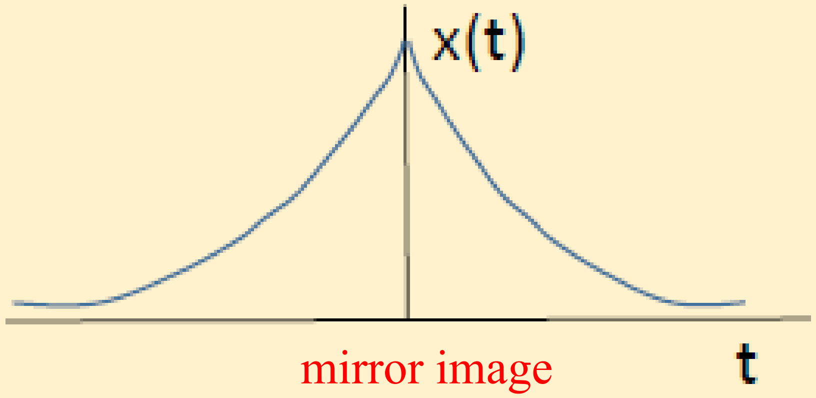
$$Even\{x_1(t)\} = \frac{x_1(t) + x_1(-t)}{2}$$

$e^{-at}u(t)$ is real; from symmetric property,

$$2\,Even\{e^{-at}u(t)\} \overset{\mathcal{F}}{\Leftrightarrow} 2\,Re\left\{\frac{1}{a + j\omega}\right\} = 2Re\left\{\frac{a - j\omega}{a^2 + \omega^2}\right\}$$

$$\Rightarrow \quad X(j\omega) = \frac{2\,a}{a^2 + \omega^2}$$

$x(t) = e^{-a|t|}$, for $a > 0$



Properties of CT Fourier Transform

4. Differentiation and Integration

$$\frac{dx(t)}{dt} \overset{\mathcal{F}}{\Longleftrightarrow} j\omega\, X(j\omega)$$

Proof

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

$$\frac{dx(t)}{dt} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\frac{d}{dt}(e^{j\omega t})d\omega$$

$$\frac{dx(t)}{dt} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)j\omega\; e^{j\omega t}d\omega$$

$$\frac{dx(t)}{dt} = \frac{1}{2\pi}\int_{-\infty}^{\infty} [j\omega\; X(j\omega)]\; e^{j\omega t}d\omega$$

Similarly,

$$\int_{-\infty}^{\infty} x(\tau)\, d\tau \overset{\mathcal{F}}{\Longleftrightarrow} \frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$$

DC or average value

Example 8

Determine the Fourier transform of the unit step function.

$$x(t) = u(t) \Rightarrow X(j\omega) = ?$$

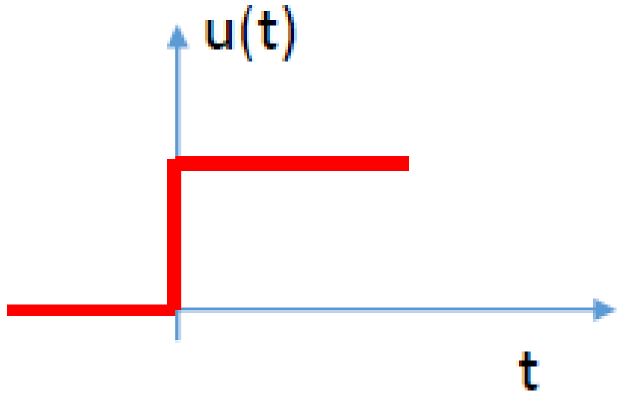
$\dfrac{du}{dt} = \delta(t)$; For unit impulse $g(t) = \delta(t) \overset{\mathcal{F}}{\Longleftrightarrow} G(j\omega) = 1$

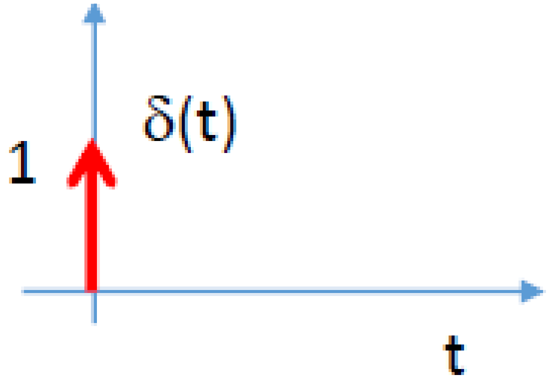
Now, $x(t) = \displaystyle\int_{-\infty}^{\infty} \delta(\tau)\, d\tau \overset{\mathcal{F}}{\Longleftrightarrow} \frac{G(j\omega)}{j\omega} + \pi G(0)\delta(\omega)$

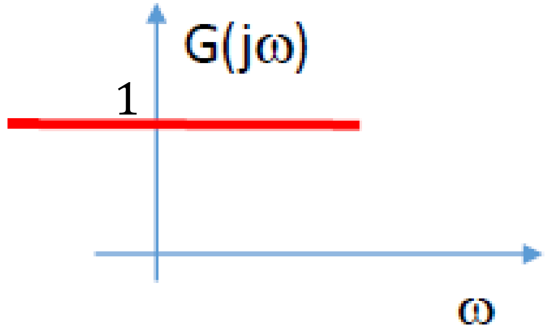
$$\Rightarrow \quad X(j\omega) = \frac{1}{j\omega} + \pi\delta(\omega) \quad \leftarrow \quad G(j\omega) = 1 \rightarrow G(0) = 1$$

Also, we observe that

$$\delta(t) = \frac{du(t)}{dt} \overset{\mathcal{F}}{\Longleftrightarrow} j\omega\left\{\frac{1}{j\omega} + \pi\delta(\omega)\right\} = 1 + \pi\, j\omega\, \cancelto{0}{\delta(\omega)} = 1 \qquad \omega\,\delta(\omega) = 0 = \begin{cases} 0 \cdot 1 & \omega = 0 \\ \omega \cdot 0 & \omega \neq 0 \end{cases}$$







Properties of CT Fourier Transform

5.Time and Frequency Scaling

If $x(t) \overset{\mathcal{F}}{\Longleftrightarrow} X(j\omega)$ Than $x(a\,t) \overset{\mathcal{F}}{\Longleftrightarrow} \dfrac{1}{|a|} X\left(\dfrac{j\omega}{a}\right)$

Proof

$$\tau = at \Rightarrow d\tau = a\,dt \Rightarrow dt = \frac{1}{a} d\tau$$

$$\mathcal{F}\{x(at)\} = \int_{-\infty}^{\infty} x(at) e^{-j\omega t} dt = \int_{-\infty}^{\infty} x(\tau) e^{-j\omega \frac{1}{a}\tau} \frac{1}{a} d\tau = \frac{1}{a}\int_{-\infty}^{\infty} x(\tau) e^{-j\frac{\omega}{a}\tau} d\tau$$

$$= \begin{cases} for\ a > 0, & \dfrac{1}{a}\displaystyle\int_{-\infty}^{\infty} x(\tau) e^{-j\frac{\omega}{a}\tau} d\tau \\ for a < 0, & \dfrac{1}{a}\displaystyle\int_{\infty}^{-\infty} x(\tau) e^{-j\frac{\omega}{a}\tau} d\tau = -\dfrac{1}{a}\displaystyle\int_{-\infty}^{\infty} x(\tau) e^{-j\frac{\omega}{a}\tau} d\tau \end{cases}$$

For $a < 0$: $t = -\infty \Rightarrow \tau = \infty; at\ \ t = \infty \Rightarrow \tau = -\infty$

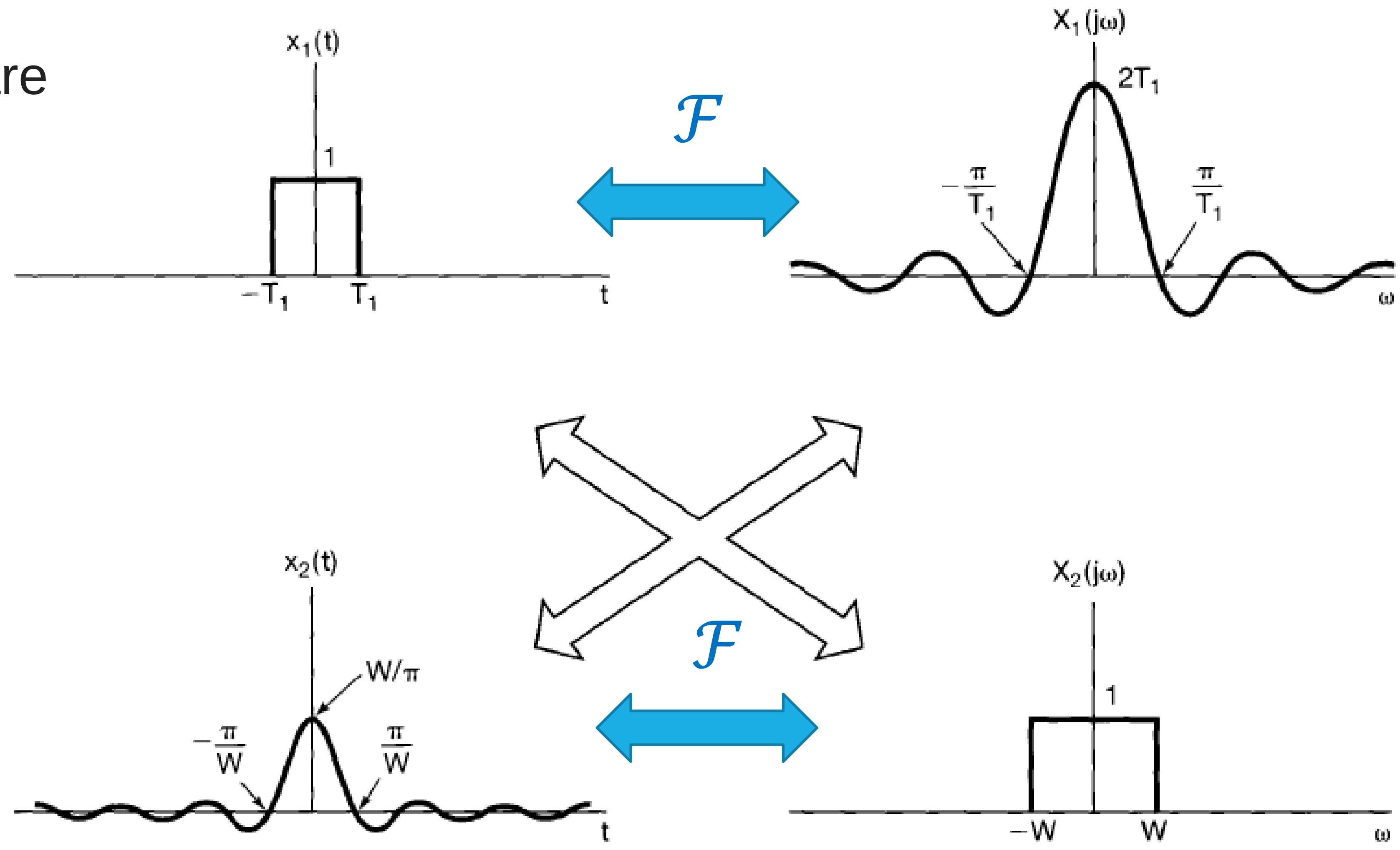
In particular, $x(-t) \overset{\mathcal{F}}{\Longleftrightarrow} X(-j\omega)$ $\Rightarrow$ Reversing a signal in time reverses its Fourier transform also.

Properties of CT Fourier Transform

6. Duality

- The FT and IFT relations are similar

- This symmetry leads to duality property of the Fourier transform.



Example 8

Use the duality property to determine $G(j\omega)$, the Fourier transform of $g(t) = 2/(1+t^2)$.

From example 2:

$$x(t) = e^{-a|t|}, a > 0 \quad \stackrel{\mathcal{F}}{\longleftrightarrow} \quad X(j\omega) = \frac{2a}{a^2+\omega^2}$$

For a = 1, $x(t) = e^{-|t|} \quad \stackrel{\mathcal{F}}{\longleftrightarrow} \quad X(j\omega) = \dfrac{2}{1+\omega^2}$

The synthesis equation for this FT pair is:

$$e^{-|t|} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{2}{1+\omega^2}\right)e^{j\omega t}d\omega \xrightarrow{\text{Multiplying by } 2\pi \text{ and replacing t by -t}} 2\pi\, e^{-|t|} = \int_{-\infty}^{\infty}\left(\frac{2}{1+\omega^2}\right)e^{-j\omega t}d\omega$$

$$\xrightarrow{\text{interchanging } t \text{ and } \omega} 2\pi\, e^{-|\omega|} = \underbrace{\int_{-\infty}^{\infty}\overbrace{\left(\frac{2}{1+t^2}\right)}^{g(t)}e^{-j\omega t}dt}_{G(j\omega)} \xrightarrow{\text{We obtain FT analysis equation}} \frac{2}{(1+t^2)} \stackrel{\mathcal{F}}{\longleftrightarrow} 2\pi\, e^{-|\omega|}$$

$$\boxed{G(j\omega) = 2\pi\, e^{-|\omega|}}$$

Convolution Property

For an LTI system: $y(t) = h(t) * x(t) \overset{\mathcal{F}}{\Longleftrightarrow} Y(j\omega) = X(j\omega)\, H(j\omega)$

A *convolution* in time domain implies a *multiplication* in Fourier domain.

Example 9

An impulse response of an LTI system: $h(t) = \delta(t - t_0)$

The frequency response of the system: $H(j\omega) = \int_{-\infty}^{\infty} \delta(t - t_0) e^{-j\omega t} dt = e^{-j\omega t_0}$

The Fourier transform of the output: $Y(j\omega) = H(j\omega) X(j\omega) = e^{-j\omega t_0} X(j\omega)$

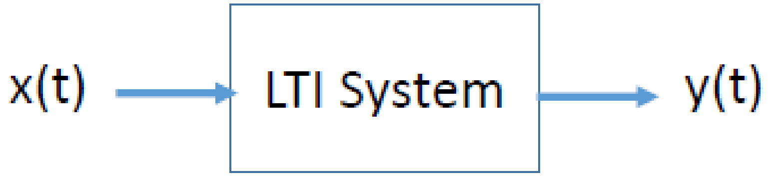
$\Rightarrow \quad y(t) = x(t - t_0)$

time shifting property.

time shifting property Slide 15.

$$Y(j\omega) = \int_{-\infty}^{\infty} y(t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} x(t - t_0) e^{-j\omega t} dt = e^{-j\omega t_0} \int_{-\infty}^{\infty} x(\tau) e^{-j\omega \tau} d\tau = e^{-j\omega t_0} X(j\omega)$$

Find $Y(j\omega)$ for the LTI systems



Example 10: Differentiator

$$y(t) = \frac{dx(t)}{dt}$$ From differential property, $$Y(j\omega) = j\omega X(j\omega)$$

This implies that $$\boxed{H(j\omega) = j\omega}$$ Frequency response of a differentiator.

Example 11: Integrator

$$y(t) = \int_{-\infty}^{t} x(\tau)d\tau$$

The impulse response of this system is a unit step, u(t).

$$h(t) = u(t) \overset{\mathcal{F}}{\Longleftrightarrow} H(j\omega) = \int_{-\infty}^{\infty} u(t)e^{-j\omega t}dt$$

From example 8 $$H(j\omega) = \frac{1}{j\omega} + \pi\delta(\omega)$$

$$\begin{aligned} Y(j\omega) &= H(j\omega)X(j\omega) \\ &= \left(\frac{1}{j\omega} + \pi\delta(\omega)\right)X(j\omega) \\ &= \frac{1}{j\omega}X(j\omega) + \pi\delta(\omega)X(j\omega) \\ &= \frac{1}{j\omega}X(j\omega) + \pi\delta(\omega)X(0) \end{aligned}$$

Example 12

Find the response, $y(t)$ of an LTI system, if $x(t)=e^{-bt}u(t),\ \ h(t)=e^{-at}u(t);\ \ a>0, b>0$

$$H(j\omega)=\int_{-\infty}^{\infty} e^{-at}u(t)e^{-j\omega t}dt=\int_{0}^{\infty} e^{-at}e^{-j\omega t}dt=\int_{0}^{\infty} e^{-(a+j\omega)t}dt=\frac{1}{a+j\omega}$$

$$X(j\omega)=\int_{-\infty}^{\infty} e^{-bt}u(t)e^{-j\omega t}dt=\frac{1}{b+j\omega}$$

$$Y(j\omega)=H(j\omega)X(j\omega)=\left(\frac{1}{a+j\omega}\right)\left(\frac{1}{b+j\omega}\right)=\frac{A}{a+j\omega}+\frac{B}{b+j\omega}$$

$$\left.\begin{aligned}&\Rightarrow 1=A(b+j\omega)+B(a+j\omega)\\&\Rightarrow 1=Ab+Ba+j\omega(A+B)\\&\Rightarrow Ab+Ba=1;\ \ A+B=0\end{aligned}\right\}$$

$$\Rightarrow\ A=\frac{1}{b-a}=-B$$

$$\Rightarrow\ Y(j\omega)=\frac{1}{b-a}\left(\frac{1}{a+j\omega}-\frac{1}{b+j\omega}\right)$$

Example 12 - contd

$$e^{-at}u(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{a+j\omega} \quad \text{and} \quad e^{-bt}u(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{b+j\omega}$$

$$\Rightarrow Y(j\omega) = \frac{1}{b-a}\left(\frac{1}{a+j\omega} - \frac{1}{b+j\omega}\right) \overset{\mathcal{F}}{\longleftrightarrow} y(t) = \frac{1}{b-a}\left(e^{-at} - e^{-bt}\right)u(t), \quad b \neq a$$

If b = a, the partial fraction expansion is not valid $\Rightarrow Y(j\omega) = \left(\frac{1}{a+j\omega}\right)\left(\frac{1}{b+j\omega}\right) = \frac{1}{(a+j\omega)^2}$

We know: $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{\left(v\frac{du}{dx} - u\frac{dv}{dx}\right)}{v^2} \Rightarrow \frac{d}{d\omega}\left(\frac{1}{a+j\omega}\right) = \frac{((a+j\omega)(0) - (1)(j))}{(a+j\omega)^2}$

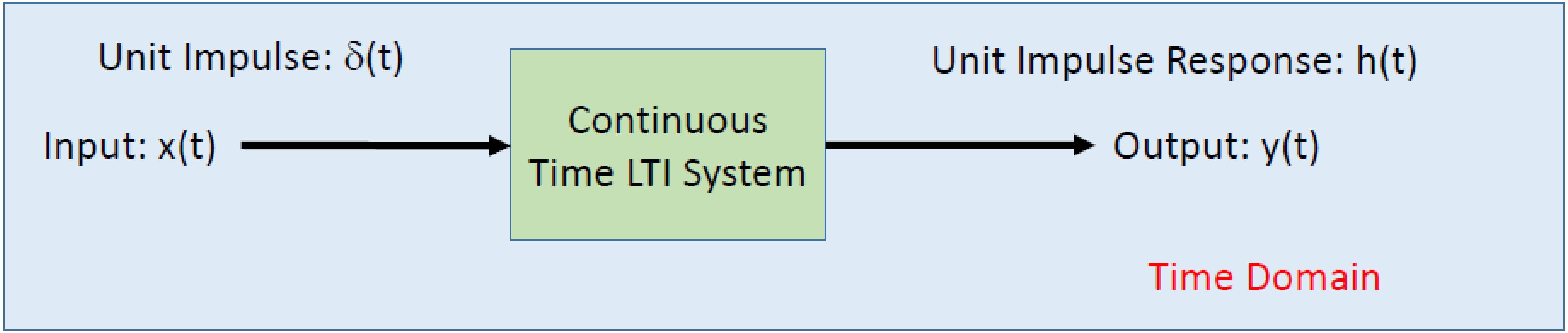
$$\Rightarrow j\frac{d}{d\omega}\left(\frac{1}{a+j\omega}\right) = \frac{1}{(a+j\omega)^2} = Y(j\omega)$$

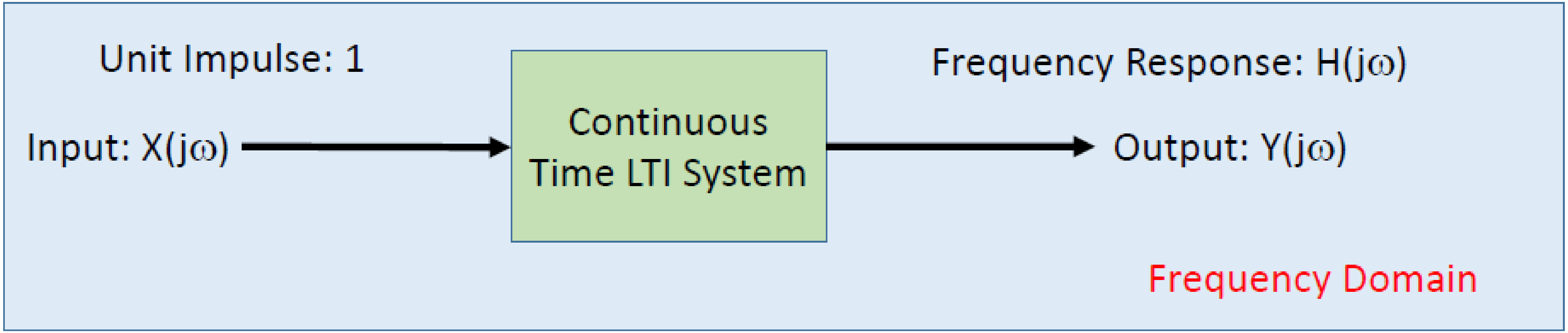
From Table 1: $tx(t) \overset{\mathcal{F}}{\longleftrightarrow} j\frac{d}{d\omega}X(j\omega) \Rightarrow$

$$t\left(e^{-at}u(t)\right) \overset{\mathcal{F}}{\longleftrightarrow} j\frac{d}{d\omega}\left(\frac{1}{a+j\omega}\right)$$

$$\Rightarrow y(t) = te^{-at}u(t), \quad a = b$$

Some Important CT Relationship





Differential Equation and Frequency Response of a System

x(t) → LTI System → y(t) $\xrightarrow{\text{differential equation}}$ $$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}.$$

The system frequency response $H(j\omega)$

$$y(t) = h(t) * x(t) \xrightarrow{\text{Convolution property,}} Y(j\omega) = H(j\omega)\,X(j\omega) \xrightarrow{\text{frequency response}} \boxed{H(j\omega) = \frac{Y(j\omega)}{X(j\omega)}}$$

Example 13

Consider the system characterized by the differential equation

$$\frac{dy(t)}{dt} + ay(t) = x(t),$$

the frequency response : $\frac{dy(t)}{dt} + ay(t) = x(t), \overset{\mathcal{F}}{\longleftrightarrow} j\omega\,Y(j\omega) + a\,Y(j\omega) = X(j\omega)$

$$(j\omega + a)\,Y(j\omega) = X(j\omega) \Longrightarrow H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{1}{j\omega + a} \xrightarrow[\text{Example 1}]{\text{impulse response}} h(t) = e^{-at}u(t)$$

Table 1:

FOURIER Transform Properties

Property	Aperiodic Signal	Fourier Transform
	$x(t)$ $y(t)$	$X(j\omega)$ $Y(j\omega)$
Linearity	$ax(t) + by(t)$	$aX(j\omega) + bY(j\omega)$
Time Shifting	$x(t - t_0)$	$e^{-j\omega t_0}X(j\omega)$
Frequency Shifting	$e^{j\omega_0 t}x(t)$	$X(j(\omega - \omega_0))$
Conjugation	$x^*(t)$	$X^*(-j\omega)$
Time Reversal	$x(-t)$	$X(-j\omega)$
Time and Frequency Scaling	$x(at)$	$\frac{1}{\|a\|}X\left(\frac{j\omega}{a}\right)$
Convolution	$x(t) * y(t)$	$X(j\omega)Y(j\omega)$
Multiplication	$x(t)y(t)$	$\frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\theta)\,Y\big(j(\omega-\theta)\big)d\theta$
Differentiation	$\frac{dx(t)}{dt}$	$j\omega X(j\omega)$
Integration	$\int_{-\infty}^{t} x(t)dt$	$\frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$
Differentiation in Frequency	$tx(t)$	$j\frac{d}{d\omega}X(j\omega)$
Conjugate Symmetry for Real Signals	$x(t)$ real	$X(j\omega) = X^*(-j\omega)$ $\mathcal{Re}\{X(j\omega)\} = \mathcal{Re}\{X(-j\omega)\}$ $\mathcal{Im}\{X(j\omega)\} = -\mathcal{Im}\{X(-j\omega)\}$ $\|X(j\omega)\| = \|X(-j\omega)\|$ $\sphericalangle X(j\omega) = -\sphericalangle X(-j\omega)$
Real and Even Signals	$x(t)$ real and even	$X(j\omega)$ real and even
Real and Odd Signals	$x(t)$ real and odd	$X(j\omega)$ purely imaginary and odd
Even-Odd Decomposition of Real Signals	$\begin{cases} x_e(t) = \mathcal{Ev}\{x(t)\}, & x(t) \text{ real} \\ x_o(t) = \mathcal{Od}\{x(t)\}, & x(t) \text{ real} \end{cases}$	$\mathcal{Re}\{X(j\omega)\}$ $j\,\mathcal{Im}\{X(j\omega)\}$

Table 2 BASIC FOURIER TRANSFORM PAIRS (Selected)

Signal	Fourier Transform
$x(t) = \begin{cases} 1, & \|t\| < T_1 \\ 0, & \|t\| > T_1 \end{cases}$	$\dfrac{2\sin(\omega T_1)}{\omega}$
$\delta(t)$	1
$u(t)$	$\dfrac{1}{j\omega} + \pi\,\delta(\omega)$
$\delta(t - t_0)$	$e^{-j\omega t_0}$
$e^{-at}\,u(t), \quad \mathcal{Re}\{a\} > 0$	$\dfrac{1}{a + j\omega}$
$te^{-at}\,u(t), \quad \mathcal{Re}\{a\} > 0$	$\dfrac{1}{(a + j\omega)^2}$
$\dfrac{t^{n-1}}{(n-1)!} e^{-at}\,u(t), \mathcal{Re}\{a\} > 0$	$\dfrac{1}{(a + j\omega)^n}$