Mathematics of Finance II: Derivative securities

MHAMED EDDAHBI
King Saud University
College of Sciences
Mathematics Department
Riyadh Saudi Arabia
e–mail: meddahbi@ksu.edu.sa

Second term 2015–2016
Chapter 1 : Futures contract

Forwards: Alternative derivation of formula
Spot transaction

- Price agreed to.
- Price paid/received.
- Item exchanged.

Prepaid forward contract

- Price agreed to.
- Price paid/received.
- Item exchanged in T–years.

Forward contract

- Price agreed to
- Price paid/received in T–years.
- Item exchanged in T–years.

A forward contract has two risks: **market risk** and **credit risk**. The market risk is related with the **volatility** of the asset price. The credit risk is related with the solvency of each party.

Futures: Definition

A future contract is a standardized agreement in which two counterparts agree to buy/sell an asset for a specified price at a specified period. The buyer in the future contract is said to be in long position (LP) on futures.

The seller in the future contract is said to be in short position (SP) on futures. The main reasons to enter into a future contract are hedging and speculation.

Difference between forwards and futures

Recall forward contracts are privately negotiated and are not standardized. Forward contracts are entirely flexible. Forward contracts are tailor–made contracts. Futures contracts are standardized instruments and FC have clearing houses that guarantee the transactions, which drastically lowers the probability of default to almost never.

The specific details concerning settlement and delivery are quite distinct

Futures contracts are **marked-to-market** daily or weekly, settlement for futures can occur over a range of dates.

A clearing house is an agency or separate corporation of a futures exchange responsible for settling trading accounts, clearing trades, collecting and maintaining margin monies, regulating delivery and reporting trading data. Clearinghouses act as third parties to all futures and options contracts - as a buyer to every clearing member seller and a seller to every clearing member buyer. Like forward contracts, futures contracts are contracts for deferred delivery. But, unlike forward contracts, futures contracts are marked to market daily or weekly. Consider "corresponding" forward and futures contracts:

- Same underlying asset.
- Delivery date in two days.
- The contracts are identical except:
- i) Forward contract is settled at maturity.
- ii) Futures contract is settled daily.
- Forward ignore taxes, transaction costs, and the treatment of margins.

Forward prices & futures prices
Example: Suppose we have for $T = 2$:
Day 0: $G(0, 0, 2) = 20$ SAR
Day 1: $G(0, 1, 2) = 10$ SAR with a 50% probability and $G(0, 1, 2) = 30$ SAR with a 50% probability
Day 2: $G(0, 2, 2) = S_2$ since the futures contract terminates.

Suppose that the interest rate is a constant 10% (effective per day).

If on day 1 $G(0, 1, 2) = 10$ SAR, the P&L of the buyer is $G(0, 1, 2) - G(0, 0, 2) = -10$ SAR. She (He) would borrow this amount at $r = 10\%$ and have to repay 11 SAR **on day 2**.

If on day 1 $G(0, 1, 2) = 30$ SAR, the P&L of the buyer is $= G(0, 1, 2) - G(0, 0, 2) = 10$ SAR. She (He) would invest this amount at $r = 10\%$ and have 11 SAR **on day 2**.

Since there is a 50% chance of paying interest of 1 SAR and a 50% chance of earning interest of 1 SAR, there is no expected benefit from marking to market **on day 1**.

Since futures contract offers no benefit as compared to the forward contract $F(0, 0, T) = G(0, 0, T)$.

Now suppose that the interest rate is not constant. Suppose that $r = 12\%$ on day 1 if $G(0,1,2) = 30$ SAR and $r = 8\%$ on day 1 if $G(0,1,2) = 10$ SAR.

If on day 1 $G(0,1,2) = 10$ SAR then the P&L of the buyer is $G(0,1,2) - G(0,0,2) = -10$ SAR. She (He) would borrow this amount at $r = 8\%$ and have to repay 10.8 SAR **on day 2**.

If on day 1 $G(0,1,2) = 30$ SAR then the P&L of the buyer is $G(0,1,2) - G(0,0,2) = 10$ SAR. She (He) would invest this amount at $r = 12\%$ and have 11.2 SAR **on day 2**.

Now there is an expected gain from marking to market $= (50\% \times 0.12 - 50\% \times 0.08) = 0.02$ SAR.

Since the futures contract offers a benefit as compared to the forward contract, $G(0,0,T)$ must exceed $F(0,0,T)$.

Now suppose that the interest rate is not constant. Suppose that $r = 8\%$ on day 1 if $G(0,1,2) = 30$ SAR and $r = 12\%$ on day 1 if $G(0,1,2) = 10$ SAR.

If on day 1 $G(0,1,2) = 10$ SAR then the P&L of the buyer is $G(0,1,2) - G(0,0,2) = -10$ SAR. She (He) would borrow this amount at $r = 12\%$ and have to repay 11.2 SAR **on day 2**.

Forward price & futures price

If on day 1 $G(0,1,2) = 30$ SAR then the P&L of the buyer is $G(0,1,2) - G(0,0,2) = 10$ SAR. She (He) would invest this amount at $r = 8\%$ and have 10.8 SAR **on day 2**.

Now there is an expected P&L from marking to market $= (50\% \times 0.08 - 50\% \times 0.12) = -0.02$ SAR.

Since the futures contract produces a loss as compared to the forward contract, $F(0,0,T)$ must exceed $G(0,0,T)$.

With this reasoning situations:

1. $G(0,0,T) = F(0,0,T)$ when interest rates are uncorrelated with the futures price.
2. $G(0,0,T) \geq F(0,0,T)$ when interest rates are positively correlated with the futures price.
3. $G(0,0,T) \leq F(0,0,T)$ when interest rates are negatively correlated with the futures price.

Stock index futures contracts

- Stock index: a weighted average of the prices of a selected number of stocks.
- Underlying: the portfolio of stocks comprising the index.

- Stock index futures contracts are heavily traded
- Examples of stock indices (futures exchanges):
  - S&P/TSX Canada 60 Index (ME)
  - S&P500 Composite Index (CME)
  - NYSE Composite Index (NYFE)

Where you buy and/or sell futures contracts
Futures are bought and sold in organized futures exchanges. The biggest future exchanges are:

- South African Futures Exchange (SAFEX)
- China Financial Futures Exchange (CFFEX)
- Shanghai Futures Exchange (SHFE)
- International Petroleum Exchange of London
- New York Mercantile Exchange
- London Metal Exchange
- Tokyo Commodity Exchange

Where you buy and/or sell futures contracts

- Hong Kong Futures Exchange (HKFE)
- Taiwan Futures Exchange (TAIFEX)
- Turkish Derivatives Exchange (TURDEX)
- Agricultural Futures Exchange of Thailand (AFET)
- Mercado Espaol de Futuros Financieros (MEFF)
- ICE Futures Europe, formerly London International Financial Futures and Options Exchange (LIFFE)

Futures Examples of underlying assets on which futures contracts are traded.

Category	**Description**
Stock index	S&P 500 index, Euro Stoxx 50 index, Nikkei 225, Dow-Jones Industrials, Dax, NASDAQ, Russell 2000, S&P Sectors (healthcare, utilities, technology)
Interest rate	30-year U.S. Treasury bond, 10-year U.S. Treasury notes, Fed funds rate, Euro-Bund, Euro-Bobl, LIBOR, Euribor
Foreign exchange	Euro, Japanese yen, British pound, Swiss franc, Australian dollar, Canadian dollar, Korean won
Commodity	Oil, natural gas, gold, silver, copper, aluminum, corn, wheat, lumber, hogs, cattle, milk

Futures transactions in the USA are regulated by the Commodity Futures Trading Commission (CFTC), an agency of the USA government. The clearinghouse matches the purchases and the sales which take place during the day. By matching trades, the clearinghouse never takes market risk because it always has offsetting positions with different counterparts. By having the clearinghouse as counterpart, an individual entering a future contract does not face the possible credit risk of its counterpart.

Determination of forward and futures prices
Notations:

1. t present time F_0 or $F(0,T)$ the delivery price.
2. T expiration date of forward delivery (maturity).
3. S_t, S_T the underlying prices at time t and T.
4. f_t the value of the forward contrat at time t.
5. $F(t,T)$ forward price at time t (which matures at T).
6. r the continuously risk-free interest rate at time t for investment of horizon T.
7. q dividend rate on the underlying

Intuitively the froward price has the form

$$F(t,T) = Fonction(t,T,r,q,S_t,K)$$

But the Fonction is unknown !

Arbitrage principle

The case where r is not random

For non-dividends paying stock. If $F(t,T) > S_t e^{r(T-t)}$ then, the arbitrageur choose the following strategy:

- Takes short position on the forward/futures at the price $F(t,T)$ for maturity T. The corresponding payoff is is $F(t,T) - S_T$,
- Borrows the amount S_t with the rate r. The corresponding payoff is given by $S_t e^{r(T-t)}$,
- Takes a short position on the stock by buying one share of the asset at the price S_t in order to compensate the risk of the forward/futures contrat. The payoff of this position is S_T when selling the share at tile T.

At maturity date T, the broker gets a net profit equal to $F(t,T) - S_t e^{r(T-t)}$.

Now, if $F(t,T) < S_t e^{r(T-t)}$ then the broker choose the following strategy:

- long a forward/futures at the price $F(t,T)$ maturing at time T. The payoff is $S_T - F(t,T)$,
- Short sell one share of the asset S_t, in order to buy it at expiration date. The payoff is $-S_T$,
- Invest in bank the amount S_t with the rate r. The payoff is $-S_t e^{r(T-t)}$

At expiration date T, the broker ends up a profit equal $S_t e^{r(T-t)} - F(t,T)$.

Therefore the fair price is then $F(t,T) = S_t e^{r(T-t)}$.

Pricing of Forwards et Futures for investment assets

Asset	**For / Fut price**	**C. V. at t**
Non-dividends	$S_0 e^{rT}$	$S_t - \dfrac{F_0}{e^{r(T-t)}}$
Return with P.V. I	$(S_0 - I)\, e^{rT}$	$S_t - I_t - \dfrac{F_0}{e^{r(T-t)}}$
Return with rate q	$S_0 e^{(r-q)T}$	$\dfrac{S_t}{e^{q(T-t)}} - \dfrac{F_0}{e^{r(T-t)}}$

where $F_0 = F(0,T)$ for ease of notations is the delivery price which is given by the second column.

Futures and forward contrats on currency
We can assimilate the foreign currencies as assets paying dividends with the rate $q = r_f$ which representes the exchange rate of the currency at time T.

$$F(t,T) = S_t e^{(r-r_f)(T-t)}$$

If this formula is not satisfied then there will be arbitrage opportunities.
Futures on consumable assets
By consumable assets we mean asset which can be consumed, such as commodities, raw materials in industrial products (gold, copper, oil, etc.) or products food (wheat, oil, cocoa, sugar, coffee, etc).

For this kind of asset we must consider the costs of carry or warehousing that includes rent. The arbitrage relationship between the futures price and the current price of the asset is given

$$F(t,T) = S_t e^{(c-y)(T-t)} \begin{cases} c = r \text{ non-dividends} \\ c = r - q \text{ with dividends} \\ c = r + u \text{ cost of carry} \end{cases}$$

where c is *le cost of carry* and y is the convenience yield. This parameter reflects the market's expectations concerning the future availability of the commodity.
Forwards: Alternative derivation of formula
Forward price when the underlying asset provides a known yield q: $F_p(0,t,T) = S_t e^{-q(T-t)}$:

$F_p(0,t,T)$ equals the investment required in the asset at time t (today) that will yield one unit of the asset in T–years when physical delivery occurs.

$e^{-q(T-t)}$ units of the asset will grow to $e^{-q(T-t)} \times e^{q(T-t)} = 1$–unit of the asset in T–years, assuming that the income provided by the asset is reinvested in the asset.

$e^{-q(T-t)}$ units of the asset cost $S_t e^{-q(T-t)}$ today (at time t).

$$F(0,t,T) = F_p(0,t,T)e^{r(T-t)} = S_t e^{(r-q)(T-t)}$$

A forward contract allows the long position to delay payment for T–years and requires the short position to delay receipt. The LP can earn interest on the cash that would otherwise have been paid. The SP foregoes this interest. The forward price (which is arrived at by multiplying the prepaid forward price, equal to $S_t e^{-q(T-t)}$ by $e^{r(T-t)}$ compensates the SP for the delay.
Futures and hedging
An airline company may want to hedge its bets against an unexpected increase in

jet fuel prices. Its traders will therefore seek to enter into a futures contract to lock in a purchase price closer to today's prices for jet fuel.

They may buy a futures contract agreeing to buy 1 million gallons of JP-8 fuel, taking delivery 90 days in the future, at a price of 3 dollars per gallon.

Someone else naturally wants to ensure they have a steady market for fuel.

They also want to protect themselves against an unexpected decline in fuel prices, so they will gladly enter into either a futures contract.

In this example, both parties are hedgers, rather than speculators.

They are turning to the futures market as a way to manage their exposure to risk, rather than make money off of the deal directly.

Futures: Arbitrage trade

There are also people who seek to make money off of changes in the price of the contract itself, when bought or sold to other investors.

Naturally, if the price of fuel rises, the contract itself becomes more valuable, and the owner of that contract could, if it chose, sell that contract for someone else who is willing to pay more for it.

It may make sense for another airline to pay 10 cents per gallon for a contract to save 20 cents. And so there is a lively and relatively liquid market for these contracts, and they are bought and sold daily on exchanges.

Example: The S&P 500 Futures Contract

Specifications for the S&P500 index futures contract	
Underlying	S&P 500 index
Where traded	Chicago Mercantile Exchange
Size	250 $\times$ S&P 500 index
Months	March, June, September, December
Trading ends	Business day prior to determination of settlement price
Settlement	Cash-settled, based up on opening price of S&P500 on third Friday of expiration month

The S&P 500 futures contract has the S&P 500 stock index as the underlying asset. Futures on individual stocks have recently begun trading in the United States. The notional value, or size, of the contract is the dollar value of the assets underlying one contract. In this case it is by definition $250\$ \times 1300 = 325,000.12$ The S&P 500 is an example of a cash-settled contract: Instead of settling by actual delivery of the underlying stocks, the contract calls for a cash payment that equals the profit or loss as if the contract were settled by delivery of the underlying asset.

On the expiration day, the S&P 500 futures contract is marked-to-market against the actual cash index. This final settlement against the cash index guarantees that the futures price equals the index value at contract expiration. It is easy to see why the S&P 500 is cash-settled. A physical settlement process would call for delivery of 500 shares (or some large subset thereof) in the precise percentage they make up the S&P 500 index. This basket of stocks would be expensive to buy and sell. Cash settlement is an inexpensive alternative.

Margins and Marking to Market

Let us explore the logistics of holding a futures position. Suppose the futures price is 1100 and you wish to acquire a 2.2 million US $ position in the S&P500 index. The notional value of one contract is $250 \times 1100 = 275000$: this represents the amount you are agreeing to pay at expiration per futures contract. To go long 2.2 million USA $ of the index, you would enter into $2.2million/0.275million = 8$ long futures contracts. The notional value of eight contracts is $8 \times 250 \times 1100 = 2000 \times 1100 = 2.2$ million $.

The margin on the S&P500 contract has generally been less than the 10% we assume in this example.

See Excel sheets for practice

Example: some common futures

1. Crude oil futures trade in units of 1000 U.S. barrels (42,000 gallons). The underlying is a US barrel. The notional amount is 1000 barrels. The current price is $70 per barrel. Hence, the current value of a future contract on crude oil is $70000.

2. S&P500 future contracts trade on 250 units of the index. They are cash settled. At expiration time, instead of a sale, one of the future counterpart receive a payment according with S&P500 spot price at expiration. The current price of S&P500 is 1500. The current value of a future contract on S&P500 is (250)(1500) = $375000.

Suppose that two parties agree in a future contact for crude oil for delivery in 18 months. The contract is worth $70000. Usually future positions are settled into the margin account either every day or every week. By every day we mean every day which the market is open. Let us suppose that a clearinghouse settles accounts daily.

Suppose that the annual continuously compounded interest rate is r. Every day, the profit or loss is calculated on the investor's futures position. If there exists a loss, the investor's broker transfers that amount from the investor's margin

account to the clearinghouse. If a profit, the clearinghouse transfers that amount to investor's broker who then deposits it into the investor's margin account. The profit for a long position in a future contract is

$$M_{t-(1/365)} \times (\exp(r/365) - 1) + N(S_t - S_{t-(1/365)}).$$

where $M_{t-(1/365)}$ is the yesterday's balance in the margin account, N is the notional amount, S_t is the current price, $S_{t-(1/365)}$ is the yesterday price. Hence, after the settlement, the balance in the investor's (buyer) margin account is

$$M_t = M_{t-(1/365)} \times \exp(r/365) + N(S_t - S_{t-(1/365)}).$$

The profit for a short position in a future contract is

$$M_{t-(1/365)} \times (1 - \exp(r/365)) + N(S_{t-(1/365)} - S_t).$$

Marking–to–market is to calculate the value of a future contract according with the current value of the asset.

On July 5, 2007, ABC enters a long future contract for 1,000 U.S. barrels of oil at \$71.6 per barrel. The margin account is 50% of the market value of the futures' underlier. The annual continuously compounded rate of return is 6%.
(i) On July 6, 2007, the price of oil is \$70.3. What is the balance in ABC's margin account after settlement?
(ii) On July 7, 2007, the price of oil is \$72.1.

What is the balance in ABC's margin account after settlement?
Solution: (i) The initial balance in ABC's margin account is $0.50 \times 1000 \times 71.6 = 35800$.

The balance in ABC's margin account on July 6, 2007, after settlement, is

$$M_{t-(1/365)} \exp(r/365) + N(S_t - S_{t-(1/365)})$$

$$= (35800) \exp(0.06/365) + (1000)(70.3 - 71.6) = 35105.89.$$

Since the price of the oil decreases, the value of having 1000 barrels in 18 months decreases.
Solution: (ii) The balance in ABC's margin account on July 6, 2007, after settlement, is

$$M_{t-(1/365)} \exp(r/365) + N(S_t - S_{t-(1/365)})$$

$$= (35105.89) \exp(0.06/365) + (1000)(72.1 - 70.3)$$

$$= 35711.56.$$

Notice that this balance is different from

$$(35800)\exp(0.06(2/365)) + (1000)(72.1 - 71.6) = 36311.77.$$

In the first day, ABC's account balance was smaller. So, ABC lost interest because the drop on price on July 6, 2007.

If the balance in the margin account falls the clearinghouse has less protection against default. Investors are required to keep the margin account to a minimum level. This level is a fraction of the initial margin. The maintenance margin is the fraction of the initial margin which participants are asked to hold in their accounts. If the balance in the margin account falls below this level, an investor's broker will require the investor to deposit funds sufficient to restore the balance to the initial margin level. Such a demand is called a margin call. If an investor fail to the deposit, the investor's broker will immediately liquidate some or all of the investor's positions. A company enters into a short futures contract to sell 100000 pounds of frozen orange juice for \$1.4 cents per pound. The initial margin is 30% and the maintenance margin is 20%. The annual effective rate of interest is 4.5%. The account is settled every week. What is the minimum next week price which would lead to a margin call?

Solution: The initial balance in the margin account is

$$(0.30) \times (100000) \times (1.4) = 42000.$$

The minimum balance in the margin account is

$$(0.20) \times (100000) \times (1.4) = 28000.$$

After settlement next week balance is

$$42000(1.045)^{1/52} + 100000(1.4 - S_{1/52}).$$

A margin call happens if

$$28000 \quad > \quad 42000(1.045)^{1/52} + 100000(1.4 - S_{1/52}),$$

or

$$S_{1/52} \quad > \quad 1.4 - \frac{28000 - 42000(1.045)^{1/52}}{100000} = 1.540355672.$$

Advantages of futures versus forwards

The two main advantages of futures versus forwards are liquidity and counter–party risk. It is much easier to cancel before expiration a future contract than a

forward contract. Since the trade is made against a clearinghouse, a participant does face credit risk. At the same time, the margin and the marking to market reduces the default risk.

Difference between forward and futures contracts

	Forward	**Futures**
Standardized contract (delivery date, quality, quantity)	No	Yes
Traded in primary market standardized exchanges	O-T-C	Yes
Credit risk (default risk)	Yes	No
Settlement	Maturity	Daily or Weekly
Clearinghouse	No	Yes
Margin requirement	No	Yes
Transaction cost	High	Low
Regulations	No	Yes

Using forward and futures for hedging

Consider a company **ABC** based in USA which importes goods from Germany

ABC	Oulay	Portefolioe	Maturity	forward price
ABC	10 Millions €	long forward	3 months	1.1503

The forward price agreed to pay 11.503 Millions \$. If at the end of 3 months the exchange spot rate is 1.1400, without hedging the outlay will be 11.400 Millions \$ which is less than 11.503 Millions \$. But if exchange spot rate is 1.17, the 10 Millions € worth 11.7 millions \$ the company ABC will regret no hedging.

Oil : Hedging a short position on the stock.

Assume that an oil producer sign a contrat with physical settlement of **1 Million** barrels of oil to deliver on July 25, 2016 with the (**spot price**). This corresponds to short the stock.

Remark that each decrease of the price by one **cent** leads to a **loss** of 10000\$.

Question how to mange or reduce this risk

Oil : Hedging a short position on the stock.

Answer : Go the Futures market (NYMEX) and short a futures contract on 1 Million barrels for July 25, 2016.

In the standards of oil each contract contains 1000 barrels.

Strategy	Spot at time 0	futures price	Spot at time T	Payoff
1000 SP	$41\$$	$39\$$	$\begin{cases} 44\$ \\ 36\$ \end{cases}$	$\begin{cases} -5\$ \\ +3\$ \end{cases}$
SP symb	S_0	F_0	S_T	$F_0 - S_T$

The payoff of the global position is $= 1\bar{M}(F_0 - S_T) + 1\bar{M}S_T = 1\bar{M}F_0$

Copper: Hedging a long position on the stock.

A company needs copper for his industry process and should buy 100000 pounds of copper on June 15, 2016. This company signs a contrat with physical delivery to buy copper from a producer but with the **spot price**. Remark that each increase of the price by one **cent** leads to a **loss** of 10000$.

Question how to mange or reduce this risk ?

Copper: Hedging a long position on the stock.

Answer : Go to the Futures market (COMEX) and long 4 futures contract on copper for June 15, 2016.

For copper each contract contains 25000 pounds.

Strategy	Spot at 0	futures price	Spot at T	Payoff
4 LP	168 cent	150 cent	$\begin{cases} 160 \text{ cent} \\ 140 \text{ cent} \end{cases}$	$\begin{cases} 10 \text{ cent} \\ -10 \text{ cent} \end{cases}$
LP symb	S_0	F_0	S_T	$S_T - F_0$

The payoff of the global position is $= 25M(S_T - F_0) - 25MS_T = -25MF_0$

Using futures or forward for arbitrage trade

Consider once again the futures contract where the underlying asset is ABC. Moreover assume that:

1. In the spot market the asset ABC is selling for 1000 SAR
2. Asset XYZ pays the holder (with certainty) 120 SAR per year in four quarterly payments of 30 SAR, and the next quarterly payment is exactly 3 months from now.
3. The futures contract requires delivery 3 months from now.
4. The current 3-month interest rate at which funds can be loaned or borrowed is 2% per year.

Suppose that an investor or a broker suggests to buy a futures contract for 1008 SAR with cash delivery in 3 months.

Consider the following strategy:

1. Short futures: Sell the futures contract at 1010 SAR.
2. Borrow 1000 for 3 months at 2% per year.
3. Purchase an unit of the asset ABC at the market for $S_0 = 1000$.

At maturity (in 3 months):

1. Sell one unit of the asset for $S_{3months}$.
2. Settle the futures contract for $1010 - S_{3months}$. This to positions leads to a cash of 1010 SAR.
3. Repay the loan which sum-up to $1000 \times (1 + 2/4 \times 0,005) = 1005$.

This strategy ends up with a profit $= 5$ SAR without risk and without initial investment