Math 423 Spring 2008 Final exam

قسم الرياضيات

جامعة الملك سعود

٤٢٣ ريض الاختبار النهائي

الفصل الثاني ٢٠٠٧ – ٢٠٠٨

١٤٢٩

س١ (أ) استنتج المعادلة التفاضلية للسطوح $u(x,y)=\sqrt{x}\, f(x+\sin y)$

حيث f دالة اختيارية.

(ب) جد الحل العام للمعادلة $u_x - u_y + u_z - u = 1$

س٢ جد الدالة التوافقية في الحلقة الدائرية $\{(r,\theta): 1<r<2, -\pi<\theta\le\pi\}$

التي تحقق $u(1,\theta)=1$ ، $u(2,\theta)=2$.

س٣ اوجد الحل للمسألة

$$u_{tt} = u_{xx} - \sin x \quad 0<x<\pi,\ t>0$$

$$u(0,t) = u(\pi,t) = 0 \quad t>0$$

$$u(x,0) = u_t(x,0) = 0 \quad 0<x<\pi$$

س٤ استخدم فصل المتغيرات لحل المعادلة $u_{tt} + 2u_t = c^2 u_{xx}$

في $(0,1)\times(0,\infty)$ تحت الشروط

$u(0,t) = u(1,t) = 0 \ \forall t>0;\ u(x,0)=$

حيث $x>0$ ، $t>0$

u_x

$u(x,$

دالة $u(x,t)$ واثبت أن $\lim_{t\to\infty} u(x,t)=0$.

صيغة الحل أن $\frac{\pi}{2}=\int_0^\infty \frac{\sin\lambda}{\lambda}\,d\lambda$.

(a) 1 $u_x = \frac{1}{2\sqrt{x}} f + \sqrt{x} f'$

1 $u_y = \sqrt{x} f' \cos y$

$\Big\} \Rightarrow \cos y\, u_x - u_y = \frac{\cos y}{2\sqrt{x}} f = \frac{\cos y}{2x} u$ 2

$2x \cos y\, u_x - 2x u_y = \cos y\, u$

(b)

1 $dx = -dy = dz = \frac{du}{u+1}$

2 $x+y = c_1, \quad y+z = c_2, \quad \ln(u+1) = x + c \Rightarrow u+1 = c_3 e^x$

$F(x+y, y+z, e^{-x}(u+1)) = 0$ or $u(x,y) = e^x f(x+y, y+z) - 1$ 1

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1 $u(r,\theta) = a_0 + d_0 \ln r + \sum_1^\infty (c_n r^n + d_n r^{-n})(a_n \cos n\theta + b_n \sin n\theta)$

$u(1,\theta) = a_0 + \sum_1^\infty (c_n + d_n)(a_n \cos n\theta + b_n \sin n\theta) = 1$

3 $\Rightarrow a_0 = 1, \quad c_n + d_n = 0 \quad \forall n \geq 1$

$u(2,\theta) = 1 + d_0 \ln 2 + \sum_1^\infty (c_n 2^n + d_n 2^{-n})(a_n \cos n\theta + b_n \sin n\theta) = 2$

3 $\Rightarrow d_0 = \frac{1}{\ln 2}(2-1) = \frac{1}{\ln 2}, \quad c_n 2^n + d_n 2^{-n} = 0 \quad \forall n \geq 1$

$c_n + d_n - 2^n (c_n 2^n + d_n 2^{-n}) = (1 - 2^{2n}) c_n = 0 \Rightarrow c_n = 0 \quad \forall n \geq 1$

$\Rightarrow d_n = 0 \quad \forall n \geq 1$

2 $\therefore u(r,\theta) = 1 + \ln r / \ln 2$

Let $u(x,t) = v(x,t) + \varphi(x)$, where $\begin{cases} v_{tt} = v_{xx} \\ v(0,t) = v(\pi,t) = 0 \\ v(x,0) = -\varphi(x), \; v_t(x,0) = 0 \end{cases}$

$\Downarrow$

$v_n(x,t) = \sin nx (a_n \cos nt + b_n \sin nt)$ 3

$\therefore \varphi''(x) = \sin x$

$\varphi(x) = -\sin x + ax + b$

$\varphi(0) = \boxed{b = 0}$

3 $\varphi(\pi) = a\pi = 0 \Rightarrow \boxed{a = 0}$

$\therefore v(x,t) = \sum_1^\infty v_n(x,t) = \sum_1^\infty \sin nx (a_n \cos nt + b_n \sin nt)$

$v(x,0) = \sum_1^\infty a_n \sin nx = \sin x \Rightarrow a_1 = 1, \; a_n = 0 \; \forall n \geq 2$ 2

$v_t(x,0) = \sum_1^\infty n b_n \sin nx = 0 \Rightarrow b_n = 0 \; \forall n$ 2

$\therefore \boxed{u(x,t) = \sin x \cos t - \sin x}$ 1

$v w'' + 2 v w' = c^2 v'' w$

$\frac{5}{4}$ (13)

Let $u(x,t) = v(x) w(t)$

2 $\frac{v''}{v} = \frac{w'' + 2w'}{c^2 w} = -\lambda^2$

$m^2 + 2m + \lambda_n^2 c^2 = 0$

(1) $v(x) = a\cos\lambda x + b\sin\lambda x$

$m = -1 \pm \sqrt{1 - n^2\pi^2 c^2} = -1 \pm i\sqrt{n^2\pi^2 c^2 - 1}$

b.c. $\Rightarrow$ $a = 0$, $\lambda = n\pi$

$= -1 \pm i\alpha_n$

2 $\therefore$ $v(x) = b_n \sin n\pi x$

(2) $w_n(t) = e^{-t}(c_n \cos\alpha_n t + d_n \sin\alpha_n t)$ 3

$w_n'(t) = -e^{-t}(c_n\cos\alpha_n t + d_n\sin\alpha_n t) + \alpha_n e^{-t}(d_n\cos\alpha_n t - c_n\sin\alpha_n t)$

i.c.(ii) $\Rightarrow$ $w_n'(0) = -c_n + \alpha_n d_n = 0$ $\Rightarrow$ $\boxed{c_n = \alpha_n d_n}$ 3

$\therefore$ $u(x,t) = \sum_1^\infty v_n(x) w_n(t) = e^{-t}\sum_{n=1} B_n \sin n\pi x\,(\alpha_n\cos\alpha_n t + \sin\alpha_n t)$

i.c.(i) $\Rightarrow$ $u(x,0) = \sum_{n=1}^\infty \alpha_n B_n \sin n\pi x = 1$

$B_n = \frac{2}{\alpha_n}\int_0^1 \sin n\pi x\,dx = -\frac{2}{n\alpha_n\pi}\cos n\pi x\Big|_0^1 = \frac{2}{n\alpha_n\pi}(1 - \cos n\pi)$ 3

$u(x,t) = e^{-t}\sum_0^\infty \frac{4}{(2n+1)\alpha_{2n+1}\pi}\sin(2n+1)\pi x\,(\alpha_{2n+1}\cos\alpha_{2n+1}t + \sin\alpha_{2n+1}t)$

$\frac{0}{4}$ (10) +(2)

$u_\lambda(x,t) = [A(\lambda)\cos\lambda x + B(\lambda)\sin\lambda x]\,e^{-k^2\lambda^2 t}$

$\frac{\partial u_\lambda}{\partial x}(0,t) = \lambda B(\lambda)\,e^{-k^2\lambda^2 t} = 0 \quad \forall t > 0$

$\Rightarrow \boxed{B(\lambda) = 0}$ 2

$u(x,t) = \int_0^\infty u_\lambda(x,t)\,d\lambda = \int_0^\infty A(\lambda)\cos\lambda x\, e^{-k^2\lambda^2 t}\,d\lambda$ 2

$u(x,0) = \int_0^\infty A(\lambda)\cos\lambda x\,d\lambda = f(x)$ 1

$\Rightarrow A(\lambda) = \frac{2}{\pi}\int_0^\infty f(y)\cos\lambda y\,dy$

$= \frac{2}{\pi}\int_0^1 \cos\lambda y\,dy$

$= \frac{2}{\lambda\pi}\sin\lambda$ 2

$u(x,t) = \frac{2}{\pi}\int_0^\infty \frac{\sin\lambda}{\lambda}\cos\lambda x\, e^{-k^2\lambda^2 t}\,d\lambda$ 1

$|u(x,t)| \le \frac{2}{\pi}\int_0^\infty \frac{|\sin\lambda|}{\lambda} e^{-k^2\lambda^2 t}\,d\lambda = \frac{2}{\pi}\left(\int_0^{\pi/2} + \int_{\pi/2}^\infty\right) e^{-k^2\lambda^2 t}\,d\lambda$ 2

$\le \frac{2}{\pi}\int_0^{\pi/2} e^{-k^2\lambda^2 t}\,d\lambda + \frac{2}{\pi}\int_{\pi/2}^\infty \frac{1}{\pi/2} e^{-k^2\lambda^2 t}\,d\lambda \to 0 + 0$

as $t \to \infty$

Set $t = x = 0$: $u(0,0) = \frac{2}{\pi}\int_0^\infty \frac{\sin\lambda}{\lambda}\,d\lambda = 1$ 2