King Saud University

College of Sciences

Department of Mathematics

151 Math Exercises

(4,1)

Relations and Their Properties

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2018

DEFINITION 1 Let A and B be sets. A *binary relation from A to B* is a subset of $A \times B$.

In other words, a binary relation from A to B is a set T of ordered pairs where the first element of each ordered pair comes from A and the second element comes from B. We use the notation $a\,T\,b$ to denote that $(a, b) \in T$ and $a \not{T} b$ to denote that $(a, b) \notin T$. Moreover, when (a, b) belongs to T, a is said to be **related to** b by T.
Binary relations represent relationships between the elements of two sets.

EXAMPLE 1 Let $A = \{0, 1, 2\}$ and $B = \{a, b\}$. Then $\{(0, a), (0, b), (1, a), (2, b)\}$ is a relation from A to B. This means, for instance, that $0T\,a$, but that $1 \not{T} b$. Relations can be represented graphically, as shown in Figure 1, using arrows to represent ordered pairs.

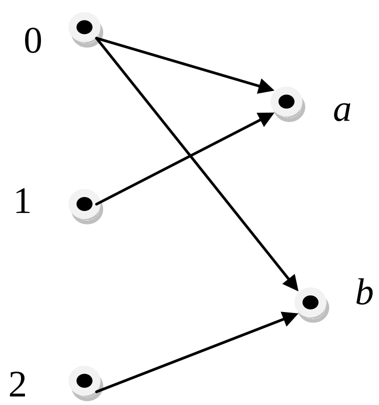


FIGURE 1 Displaying the Ordered Pairs in the Relation *T*

Relations on a Set

Relations from a set A to itself are of special interest.

DEFINITION 2 A *relation on a set A* is a relation from A to A. In other words, a relation on a set A is a subset of $A \times A$.

EXAMPLE 2 Let A be the set {1, 2, 3, 4}. Which ordered pairs are in the relation

$$R = \{(a, b) \mid a \text{ divides } b\}?$$

Solution: Because (a, b) is in R if and only if a and b are positive integers not exceeding 4 such that a divides b, we see that

$R = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)\}$.

The pairs in this relation are displayed graphically form in Figure 2.

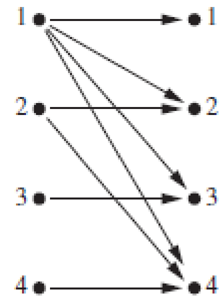
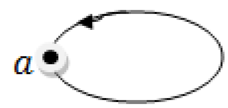


FIGURE 2 Displaying the Ordered Pairs in the Relation R from Example 2.

Properties of Relations

DEFINITION 3 A relation R on a set A is called *reflexive* if $(a, a) \in R$ for every element $a \in A$. $\forall\, a \in A\ , aRa$



EXAMPLE 3 $R = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\}$ is *reflexive*

DEFINITION 4 A relation R on a set A is called *symmetric* if $(b, a) \in R$ whenever $(a, b) \in R$, for all $a, b \in A$. $\forall a, b \in A\ ,\ aRb \Rightarrow bRa$

EXAMPLE 4 $R = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\}$ is *symmetric*

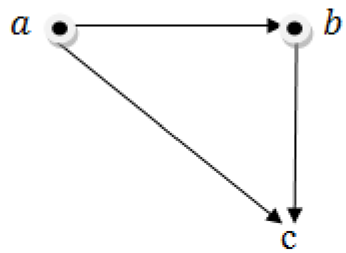
DEFINITION 5 A relation R on a set A such that for all $a, b \in A$, if $(a, b) \in R$ and $(b, a) \in R$, then $a = b$ is called *antisymmetric*.

EXAMPLE 5 $a \le b \ \wedge\ b \le a \ \Rightarrow\ a = b \ : a, b \in A \quad \therefore \le$ is *antisymmetric*.

DEFINITION 6 A relation R on a set A is called *transitive* if whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$, for all $a, b, c \in A$.

$$aRb\ \&\ bRc \ \Rightarrow\ aRc$$

$$(a,b) \in R \ \& \ (b,c) \in R \ \Rightarrow \ (a,c) \in R \ , \forall \ a,b,c \in A$$



EXAMPLE 6 $a|b \ \wedge \ b|c \ \Rightarrow \ a|c$, $\therefore |$ is *transitive*

Combining Relations

EXAMPLE 7 Let $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4\}$.

The relations $R_1 = \{(1, 1), (2, 2), (3, 3)\}$ and $R_2 = \{(1, 1), (1, 2), (1, 3), (1, 4)\}$ can be combined to obtain

$R_1 \cup R_2 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (3, 3)\}$,

$R_1 \cap R_2 = \{(1, 1)\}$,

$R_1 - R_2 = \{(2, 2), (3, 3)\}$,

$R_2 - R_1 = \{(1, 2), (1, 3), (1, 4)\}$.

$R_1 \oplus R_2 = R_1 \cup R_2 - R_1 \cap R_2 = \{ (1, 2), (1, 3), (1, 4), (2, 2), (3, 3) \}$

DEFINITION 7 Let R be a relation from a set A to a set B and S a relation from B to a set C. The *composite* of R and S is the relation consisting of ordered pairs (a, c), where $a \in A$, $c \in C$, and for which there exists an element $b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$. We denote the composite of R and S by $S \circ R$.

EXAMPLE 8 What is the composite of the relations R and S, where R is the relation from $\{1, 2, 3\}$ to $\{1, 2, 3, 4\}$ with $R = \{(1, 1), (1, 4), (2, 3), (3, 1), (3, 4)\}$ and S is the relation from $\{1, 2, 3, 4\}$ to $\{0, 1, 2\}$ with $S = \{(1, 0), (2, 0), (3, 1), (3, 2), (4, 1)\}$?

Solution: $S \circ R$ is constructed using all ordered pairs in R and ordered pairs in S, where the second element of the ordered pair in R agrees with the first element of the ordered pair in S. For example, the ordered pairs $(2, 3)$ in R and $(3, 1)$ in S produce the ordered pair $(2, 1)$ in $S \circ R$. Computing all the ordered pairs in the composite, we find

$$S \circ R = \{(1, 0), (1, 1), (2, 1), (2, 2), (3, 0), (3, 1)\}.$$

DEFINITION 8 Let R be a relation on the set A. The powers $R^n, n = 1, 2, 3, \ldots,$ are defined recursively by $R^1 = R$ and $R^{n+1} = R^n \circ R$.

The definition shows that $R^2 = R \circ R$, $R^3 = R^2 \circ R = (R \circ R) \circ R$, and so on.

EXAMPLE 9 Let $R = \{(1, 1), (2, 1), (3, 2), (4, 3)\}$. Find the powers R_n, $n = 2, 3, 4, \ldots$.

Solution: Because $R^2 = R \circ R$, we find that $R^2 = \{(1, 1), (2, 1), (3, 1), (4, 2)\}$.

Furthermore, Because $R^3 = R^2 \circ R$, $R^3 = \{(1, 1), (2, 1), (3, 1), (4, 1)\}$.

Additional computation $R^4 = R^3 \circ R$, so $R^4 = \{(1, 1), (2, 1), (3, 1), (4, 1)\}$.

It also follows that $R^n = R^3$ for $n = 5, 6, 7, \ldots$. The reader should verify this.

THEOREM 1

The relation R on a set A is *transitive* if and only if $R_n \subseteq R$ for $n = 1, 2, 3, \ldots$

DEFINITION 9 Let A and B be sets. R is a *binary relation from A to B* : $R \subseteq A \times B$.

Domain of R *is* $D_R = \{a : a \in A \land \exists b \in B\ (aRb)\}$, $D_R \subseteq A$

Range of R *is* $Im\, R = \{b : b \in B \land \exists a \in A\ (aRb)\}$, $Im\, R \subseteq B$

EXAMPLE 10 Let $A = \{0, 1, 2, 3\}$ and $B = \{a, b, c\}$.

$R = \{(0, a), (0, b), (1, a), (2, b)\}$ is a relation from A to B.

$$D_R = \{0, 1, 2\} \subseteq A$$

$$Im\, R = \{a, b\} \subseteq B$$

Representing Relations Using Matrices

EXAMPLE 11 Suppose that $A = \{1, 2, 3\}$ and $B = \{1, 2\}$. Let R be the relation from A to B containing (a, b), Where $a \in A, b \in B$,

$$R = \{(2, 1), (3, 1), (3, 2)\}$$

What is the matrix representing R ?

Solution: $$\mathbf{M}_R = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$

The 1s in M_R show that the pairs (2, 1), (3, 1), and (3, 2) belong to R. The 0s show that no other pairs belong to R.

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EXAMPLE 12 Let $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3, b_4, b_5\}$. Which ordered pairs are in the relation R represented by the matrix :

$$\mathbf{M}_R = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix} \quad \textbf{?}$$

Solution: Because R consists of those ordered pairs (a_i, b_j) with $m_{ij} = 1$, it follows that

$$R = \{(a_1, b_2), (a_2, b_1), (a_2, b_3), (a_2, b_4), (a_3, b_1), (a_3, b_3), (a_3, b_5)\}.$$

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The matrix of a relation on a set, which is a square matrix, can be used to determine whether the relation has certain properties. Recall that a relation R on A is reflexive if $(a, a) \in R$ whenever $a \in A$.

Thus, R is reflexive if and only if $(a_i, a_i) \in R$ for $i = 1, 2, \ldots, n$. Hence, R is reflexive if and only if $m_{ii} = 1$, for $i = 1, 2, \ldots, n$. In other words, R is reflexive if all the elements on the main diagonal of $\mathbf{M}_R$ are equal to 1, as shown in Figure 1. Note that the elements off the main diagonal can be either 0 or 1.

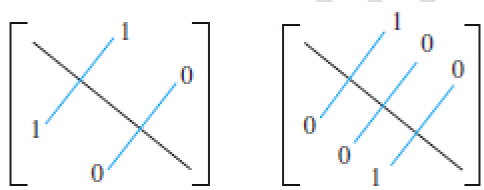
$$\begin{bmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & 1 & & & & \\ & & & \ddots & & & \\ & & & & & 1 & \\ & & & & & & 1 \end{bmatrix}$$

FIGURE 3 The Zero-One Matrix for a Reflexive Relation. (Off Diagonal Elements Can Be 0 or 1.)

The relation R is symmetric if $(a, b) \in R \Rightarrow (b, a) \in R$. Consequently, the relation R on the set $A = \{a_1, a_2, \ldots, a_n\}$ is symmetric if and only if $(a_j, a_i) \in R$ whenever $(a_i, a_j) \in R$. In terms of the entries of $\mathbf{M}_R$, R is symmetric if and only if $m_{ji} = 1$ whenever $m_{ij} = 1$. This also means $m_{ji} = 0$ whenever $m_{ij} = 0$. Consequently, R is symmetric if and only if $m_{ij} = m_{ji}$, for all pairs of integers i and j with $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, n$.

R is symmetric if and only if $\mathbf{M}_R = (\mathbf{M}_R)^t$, that is, if $\mathbf{M}_R$ is a symmetric matrix. The form of the matrix for a symmetric relation is illustrated in Figure 3(a).

The relation R is antisymmetric if and only if $(a, b) \in R$ and $(b, a) \in R \Rightarrow a = b$. Consequently, the matrix of an antisymmetric relation has the property that if $m_{ij} = 1$ with $i \neq j$, then $m_{ji} = 0$. Or, in other words, either $m_{ij} = 0$ or $m_{ji} = 0$ when $i \neq j$. The form of the matrix for an antisymmetric relation is illustrated in Figure 3(b).



(a) Symmetric (b) Antisymmetric

FIGURE 4 The Zero–One Matrices for Symmetric and Antisymmetric Relations.

EXAMPLE 13 Suppose that the relation R on a set is represented by the matrix

$$\mathbf{M}_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Is R reflexive, symmetric, and/or antisymmetric?

Solution: Because all the diagonal elements of this matrix are equal to 1, R is reflexive. Moreover, because $\mathbf{M}_R$ is symmetric, it follows that R is symmetric. It is also easy to see that R is not antisymmetric.

Suppose that R_1 and R_2 are relations on a set A represented by the matrices $\mathbf{M}_{R_1}$ and $\mathbf{M}_{R_2}$, respectively. The matrix representing the union of these relations has a 1 in the positions where either $\mathbf{M}_{R_1}$ or $\mathbf{M}_{R_2}$ has a 1. The matrix representing the intersection of these relations has a 1 in the positions where both $\mathbf{M}_{R_1}$ and $\mathbf{M}_{R_2}$ have a 1.
Thus, the matrices representing the union and intersection of these relations are

$$\mathbf{M}_{R_1 \cup R_2} = \mathbf{M}_{R_1} \vee \mathbf{M}_{R_2} \quad \text{and} \quad \mathbf{M}_{R_1 \cap R_2} = \mathbf{M}_{R_1} \wedge \mathbf{M}_{R_2}$$

EXAMPLE 14 Suppose that the relations R_1 and R_2 on a set A are represented by the matrices $\mathbf{M}_{R_1} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ and $\mathbf{M}_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

What are the matrices representing $R_1 \cup R_2$ and $R_1 \cap R_2$?

Solution: The matrices of these relations are

$$\mathbf{M}_{R_1 \cup R_2} = \mathbf{M}_{R_1} \vee \mathbf{M}_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$\mathbf{M}_{R_1 \cap R_2} = \mathbf{M}_{R_1} \wedge \mathbf{M}_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

EXAMPLE 15 Find the matrix representing the relations $S \circ R$, where the matrices representing R and S are

$$\mathbf{M}_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{M}_S = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

Solution: The matrix for $S \circ R$ is

$$\mathbf{M}_{S \circ R} = \mathbf{M}_R \odot \mathbf{M}_S = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

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The matrix representing the composite of two relations can be used to find the matrix for $\mathbf{M}_{R^n}$. In particular,

$$\mathbf{M}_{R^n} = \mathbf{M}_R^{[\boldsymbol{n}]}$$

EXAMPLE 16 Find the matrix representing the relation R^2, where the matrix representing R is

$$\mathbf{M}_R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Solution: The matrix for R^2 is

$$\mathbf{M}_{R^2} = \mathbf{M}_R^{[2]} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \odot \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

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Representing Relations Using Digraphs

DEFINITION 10 A *directed graph*, or *digraph*, consists of a set V of *vertices* (or *nodes*) together with a set E of ordered pairs of elements of V called *edges* (or *arcs*). The vertex a is called the *initial vertex* of the edge (a, b), and the vertex b is called the terminal vertex of this edge.

EXAMPLE 17 The directed graph with vertices a, b, c, and d, and edges (a, b), (a, d), (b, b), (b, d), (c, a), (c, b), and (d, b) is displayed in Figure 5.

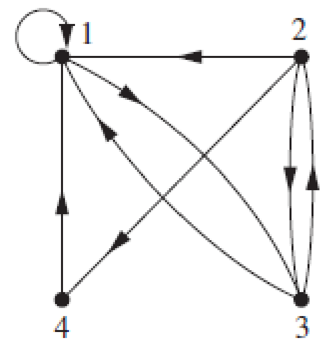


FIGURE 5 **A Directed Graph.**

EXAMPLE 18 The directed graph of the relation

R = {(1, 1), (1, 3), (2, 1), (2, 3), (2, 4), (3, 1), (3, 2), (4, 1)}

on the set {1, 2, 3, 4} is shown in Figure 6.

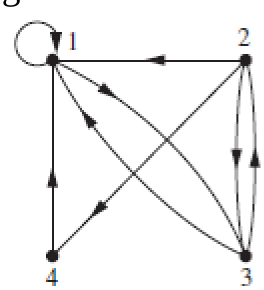


FIGURE 6 **A Directed Graph of the relation R .**

EXAMPLE 19 What are the ordered pairs in the relation R represented by the directed graph shown in Figure 7?

Solution: The ordered pairs (x, y) in the relation are

$$R = \{(1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (3, 1), (3, 3), (4, 1), (4, 3)\}.$$

Each of these pairs corresponds to an edge of the directed graph, with (2, 2) and (3, 3) corresponding to loops.

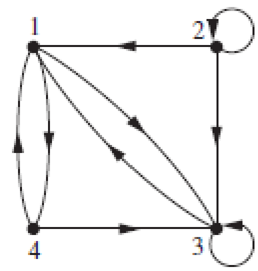


FIGURE 7 **A Directed Graph of the relation R** #

EXAMPLE 20 Determine whether the relation for the directed graphs shown in Figure 8 is *reflexive*, *symmetric*, *antisymmetric*, and/or *transitive*.

Solution: Because there are loops at every vertex of the directed graph of R, it is *reflexive*. R is *neither symmetric* nor antisymmetric because there is an edge from a to b but not one from b to a, but there are edges in both directions connecting b and c. Finally, R is *not transitive* because there is an edge from a to b and an edge from b to c, but no edge from a to c.

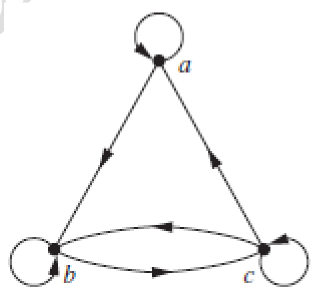


FIGURE 8 **A Directed Graph of the relation R** #

EXAMPLE 21 Determine whether the relation for the directed graphs shown in Figure 9 is *reflexive*, *symmetric*, *antisymmetric*, and/or *transitive*.

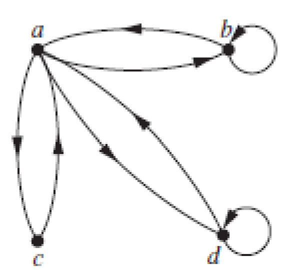


FIGURE 9 **A Directed Graph of the relation R**

Solution: Because loops are not present at all the vertices of the directed graph of S, this relation is *not reflexive.*
It is *symmetric* and *not antisymmetric*, because every edge between distinct vertices is accompanied by an edge in the opposite direction. It is also not hard to see from the directed graph that S is *not transitive*, because (c, a) and (a, b) belong to S, but (c, b) does not belong to S.

EXERCISES

1. Let R be a relation defined on the set $A = \{-2,-1,0,1,2,3,4\}$

$$m, n \in A \quad , \quad m\,R\,n \iff n = m^2$$

(i) What are the ordered pairs in the relation R ?
(ii) Find the domain and the image of R ?
(iii) Represent R by the directed graph (diagraph) ?

Solution:

2. Let R be a relation defined on the set $A = \{1,2,3,4,5\}$

$$x, y \in A, \qquad x\,R\,y \Leftrightarrow xy \leq 9$$

(i) What are the ordered pairs in the relation R ?

(ii) Find the domain and the image of R ?

(iii) Draw the directed graph (diagraph) that represents R ?

Solution:

3. Let R be a relation defined on the set $A = \{-4,-3,-2,-1,0,1,2,3,4\}$

$$x, y \in A, \qquad x\,R\,y \Leftrightarrow y = 2x$$

(i) What are the ordered pairs in the relation R ?

(ii) Find the domain and the image of R ?

Solution:

4. Let R be a relation defined from the set $A = \{1,2,3,4\}$ to the set $B = \{2,3,4,5\}$ as: $a\ R\ b \iff a + b = 5$

(i) What are the ordered pairs in the relation R ?

(ii) Find the domain and the image of R ?

(iii) Represent R with a matrix ?

Solution:

5. Suppose R is a relation defined on the set $A = \{-2, -1, 0, 1, 2\}$, as

$x, y \in A, \quad x\ R\ y \Leftrightarrow |x - y| < 2$

(i) What are the ordered pairs in the relation R ?

(ii) Draw the directed graph (diagraph) that represents R

(iii) Determine whether the relation R is *reflexive*, *symmetric*, *antisymmetric*, and/or *transitive*.

Solution:

6. Let R be a relation defined on the set $A = \{1,3,4,6\}$

$$a, b \in A, \qquad a\,R\,b \Leftrightarrow a - b = 1$$

(i) What are the ordered pairs in the relation R ?

(ii) Find the domain and the image of R ?

(iii) Draw the directed graph (diagraph) that represents R ?

Solution:

7. Let R be a relation defined on the set $A = \{0,1,2,3\}$

$$a, b \in A , \qquad a\, R\, b \Leftrightarrow a + b = 4$$

(i) What are the ordered pairs in the relation R ?

(ii) Find the domain and the image of R ?

(iii) Draw the directed graph (diagraph) that represents R ?

Solution:

8. Let R be a relation defined on the set $A = \{2,3,4,5,6\}$

$$a, b \in A, \qquad a\ R\ b \Leftrightarrow a.b < 10$$

(i) What are the ordered pairs in the relation R ?

(ii) Find the domain and the image of R ?

(iii) Draw the directed graph (diagraph) that represents R ?

(iv) Find $\mathbf{M}_R$.

Solution:

9. Let R be a relation defined on the set $A = \{-2, -1, 0, 1, 2, 3, 4\}$

$$a, b \in A, \qquad a\,R\,b \Leftrightarrow a^2 = b^2$$

(i) What are the ordered pairs in the relation R ?

(ii) Find the domain and the image of R ?

(iii) Draw the directed graph (diagraph) that represents R ?

(iv) Determine whether the relation R is *reflexive*, *symmetric*, *antisymmetric*, and/or *transitive*.

Solution:

10. Let R be a relation defined on the set $A = \{-2, -1, 0, 1, 2\}$

$$a, b \in A , \qquad a \, R \, b \Leftrightarrow a.b < 0$$

(i) What are the ordered pairs in the relation R ?

(ii) Find the domain and the image of R ?

(iii) Draw the directed graph (diagraph) that represents R ?

(iv) Find R^2 .

(v) Determine whether the relation R is *reflexive*, *symmetric*, *antisymmetric*, and/or *transitive*.

Solution:

11. Let R be a relation defined on the set $A = \{-2, -1, 0, 1, 2\}$

$$a, b \in A\ , \qquad a\ R\ b \Leftrightarrow a.b \geq 2$$

(i) What are the ordered pairs in the relation R ?

(ii) Find the domain and the image of R ?

(iii) Draw the directed graph (diagraph) that represents R ?

(iv) Find R^2 .

(v) Determine whether the relation R is *reflexive*, *symmetric*, *antisymmetric*, and/or *transitive*.

Solution:

12. Let $S = \{(1,1),(1,2),(1,3),(2,2),(3,1),(3,3)\}$ be a relation on the set $B = \{1, 2, 3\}$

(i) Draw the directed graph (diagraph) that represents S ?

(ii) Find S^2 , $\bar{S}$, $S o \bar{S}$

(iii) Find $\mathbf{M}_S$

(iv) Determine whether the relation S is *reflexive*, *symmetric*, *antisymmetric*, and/or *transitive*

Solution:

13. Let $S = \{(a,c),(b,a),(c,b)\}$ be a relation on the set $B = \{a,b,c\}$.

(i) Find $\mathbf{M}_S$?

(ii) Find $\overline{S} - S^{-1}$

(iii) Find S^2 , S^3

Solution:

14. Let $S = \{(a,b),(b,c),(c,d),(d,a)\}$ be a relation on the set $B = \{a,b,c,d\}$.

(i) Find $\mathbf{M}_S$?

(ii) Find S^2

(iii) Find $S^{-1} \circ S$

Solution:

15. Let $S = \{(1,v),(1,w),(2,u),(2,v),(3,w)\}$ and $T = \{(1,u),(1,w),(2,v),(2,w),(3,u),(3,v)\}$ are relations from the set $A = \{1,2,3\}$ to the set $B = \{u,v,w\}$.

(i) Find $\bar{S}$, $\bar{S} \cap T$, $T - \bar{S}$

(ii) Find $T^{-1} o\, S$

(iii) Find $S^{-1} \circ T$

Solution:

16. Let $R = \{(a,c),(a,b),(b,b)\}$ and
$S = \{(a,a),(a,c),(b,c),(c,a)\}$
are relations on the set $A = \{a\ ,b\ ,c\ \}$

(i) Find $(\ R\ o\ S\ ) \cap R^{-1}$

(ii) Find $S^{-1} \circ R$

(iii) Find $\mathbf{M}_R$, $\mathbf{M}_S$, $\mathbf{M}_{R\cup S}$, $\mathbf{M}_{R\cap S}$, $\mathbf{M}_{R\circ S}$

Solution:

17. Let $T = \{(1,2),(1,3),(2,2),(2,3)\}$ and $S = \{(1,1),(1,3),(2,1),(3,2)\}$ are relations on the set $E = \{1\ ,2\ ,3\ \}$

(i) Find $T\ o\ S$, $\bar{T}\ \cap S$, $\bar{T}\ o\ \bar{S}$, $T^2\ o\ S^{-1}$

(ii) Find $\mathbf{M}_T$, $\mathbf{M}_S$, $\mathbf{M}_{T\cup S}$, $\mathbf{M}_{T\cap S}$, $\mathbf{M}_{T\circ S}$

Solution:

18. Let $R = \{(a,c),(b,a),(b,b)\}$ and $S = \{(a,b),(b,b),(c,a)\}$ are relations on the set $A = \{a\ ,b\ ,c\ \}$

(i) Find $R^{-1}\ o\ S^{-1}$, $\overline{R}\ \cap\ S$, $R^2\ o\ S$

(ii) Find $\mathbf{M}_R$, $\mathbf{M}_S$, $\mathbf{M}_{R\cup S}$, $\mathbf{M}_{R\cap S}$, $\mathbf{M}_{R\circ S}$

Solution:

19. Suppose R is a relation defined on the integers set $\mathbb{Z}^{+} = \{1,2,3,\ldots\}$

$$m,n \in \mathbb{Z}^{+}, \quad m\,R\,n \Leftrightarrow m+n=20$$

Determine whether the relation T is *reflexive*, *symmetric*, *antisymmetric*, and/or *transitive*.

Solution:

1- $\because\ 5+5 \neq 20 \Rightarrow (5,5) \notin R \Rightarrow \therefore\ R$ is $irreflexive$.

2- $m,n \in \mathbb{Z}^{+}\ ,\ m\,R\,n \Leftrightarrow m+n=20$

$\xRightarrow{\text{comutative}} \quad n+m=20 \Rightarrow \therefore\ n\,R\,m \Rightarrow \therefore\ R$ is symmetric

3- $\because\ 7\,R\,13 : 7+13=20 \quad \wedge \quad 13\,R\,7 : 13+7=20$

But $7 \neq 13 \Rightarrow \therefore\ R$ is not antisymmetric

4- $\because\ 8\,R\,12 : 8+12=20 \quad \wedge \quad 12\,R\,8 : 12+7=20$

$but\ (8,8) \notin R : 8+8=16 \neq 20 \Rightarrow \therefore\ R$ is not transitive.

$\therefore\ R$ is symmetric only .

20. Suppose T is a relation defined on the integers set $\mathbb{Z}$

$$m, n \in \mathbb{Z}, \quad m \, T \, n \iff m + n \geq 2$$

Determine whether the relation T is *reflexive*, *symmetric*, *antisymmetric*, and/or *transitive*.

Solution:

21. Let T be a relation defined on the set $\mathbb{N} = \{\,1\,,2\,,3,\ldots\}$,

$m, n \in \mathbb{N}\,,\quad m\,T\,n \Leftrightarrow m + n > 3$

Determine whether the relation R is *reflexive*, *symmetric*, *antisymmetric*, and/or *transitive*.

Solution:

22. Let T be a relation defined on the integers set $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$

$$m, n \in \mathbb{Z}, \quad m\,T\,n \Leftrightarrow m + n \quad is\ odd$$

Determine whether the relation T is *reflexive*, *symmetric*, *antisymmetric*, and/or *transitive*.

Solution:

23. Let R be a relation defined on the integers set $\mathbb{N} = \{\ 1,2,3, \dots\}$

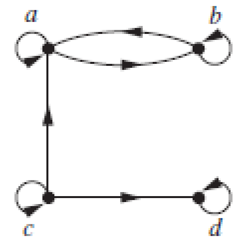
$$x, y \in \mathbb{N}\ , \quad x\ R\ y \iff x < y$$

Determine whether the relation R is *reflexive*, *symmetric*, *antisymmetric*, and/or *transitive*.

Solution:

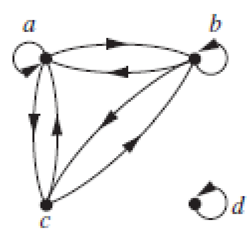
24. Determine whether the relation for the directed graphs shown in the Figure is *reflexive*, *symmetric*, *antisymmetric*, and/or *transitive*.

Solution:



25. Determine whether the relation for the directed graphs shown in the Figure is *reflexive*, *symmetric*, *antisymmetric*, and/or *transitive*.

Solution:



26. Let R be the relation represented by the matrix

$$\mathbf{M}_R = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Find the matrix representing: a) R^{-1} b) $\bar{R}$ c) R^2

Solution:

27. Let R_1 and R_2 are relations on a set A represented by the matrices

$$\mathbf{M}_{R_1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{M}_{R_2} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Find the matrices representing

a) $R_1 \cup R_2$ b) $R_1 \cap R_2$ c) $R_1 \circ R_2$ d) $R_2 \circ R_1$

Solution:

28. Let R be the relation represented by the matrix

$$\mathbf{M}_R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Find the matrix representing:

a) R^2 b) R^3 c) R^4

Solution: