Relativistic Energy

Again, let's begin with classical concepts.
The differential work done is:

$$dW = F\,dx = \frac{dp}{dt}dx$$

Dividing by dt:

$$\frac{dW}{dt} = \frac{dp}{dt}\frac{dx}{dt} = \frac{dp}{dt}\mathrm{v}$$

In terms of velocity derivatives:

$$\frac{dW}{d\mathrm{v}}\frac{d\mathrm{v}}{dt} = \frac{dp}{d\mathrm{v}}\frac{d\mathrm{v}}{dt}\mathrm{v}$$

Canceling the $d\mathrm{v}/dt$'s:

$$\frac{dW}{d\mathrm{v}} = \frac{dp}{d\mathrm{v}}\mathrm{v} \quad \text{or} \quad dW = \frac{dp}{d\mathrm{v}}\mathrm{v}\,d\mathrm{v}$$

The kinetic energy will be equal to the work done starting with zero energy and ending with W_0, or from zero velocity to u:

$$K = \int_0^{W_0} dW$$

$$= \int_0^u \frac{dp}{d\mathrm{v}}\mathrm{v}\,d\mathrm{v}$$

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Relativistic Energy

Integrating by parts: $K = \int_0^u \frac{dp}{d\mathrm{v}}\mathrm{v}\,d\mathrm{v} = p\mathrm{v}\Big|_0^u - \int_0^u p\,d\mathrm{v}$

$$= pu - m\int_0^u \frac{\mathrm{v}}{\sqrt{1-\mathrm{v}^2/c^2}}d\mathrm{v} = pu + mc^2\sqrt{1-\mathrm{v}^2/c^2}\Big|_0^u$$

substituting for p

because: $\frac{d}{d\mathrm{v}}\left[-c^2\sqrt{1-\mathrm{v}^2/c^2}\right] = -\frac{1}{2}c^2\frac{-2\mathrm{v}/c^2}{\sqrt{1-\mathrm{v}^2/c^2}}$

$$= \left(m\frac{u}{\sqrt{1-u^2/c^2}}\right)u + mc^2\left(\sqrt{1-u^2/c^2} - 1\right)$$

$$= mc^2\left(\frac{u^2/c^2}{\sqrt{1-u^2/c^2}} + \frac{1-u^2/c^2}{\sqrt{1-u^2/c^2}} - 1\right)$$

$$K = mc^2(\gamma - 1)$$

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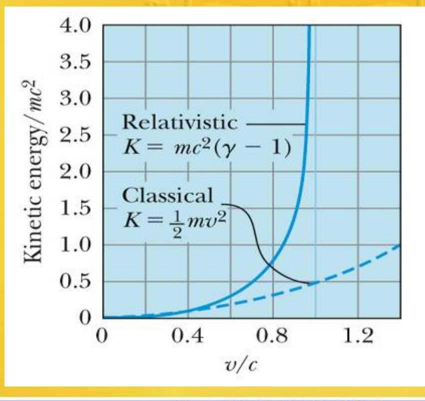
Relativistic Energy

$$K = mc^2(\gamma - 1)$$

Written in terms of $u = \text{v} \ll c$:

$$K \approx mc^2\left[\left(1 + \tfrac{1}{2}\text{v}^2/c^2\right) - 1\right] = \tfrac{1}{2}m\text{v}^2$$

the classical result!



Note that even an infinite amount of energy is not enough to achieve c.

Total Energy and Rest Energy

Manipulate the energy equation: $K = mc^2(\gamma - 1)$

$$K = \gamma mc^2 - mc^2$$

The term mc^2 is called the **Rest Energy** and is denoted by E_0:

$$E_0 = mc^2$$

The sum of the kinetic and rest energies is the **total energy** of the particle E and is given by:

$$E = \gamma mc^2 = \frac{mc^2}{\sqrt{1 - u^2/c^2}} = \frac{E_0}{\sqrt{1 - u^2/c^2}} = K + E_0$$

Momentum and Energy

Square the momentum equation, $p = \gamma mu$, and multiply by c^2:

$$p^2c^2 = \gamma^2 m^2 u^2 c^2$$

Substituting for u^2 using $\beta^2 = u^2/c^2$:

$$= \gamma^2 m^2 c^4 \left(\frac{u^2}{c^2}\right) = \gamma^2 m^2 c^4 \beta^2$$

But $\beta^2 = 1 - \frac{1}{\gamma^2}$

$$p^2c^2 = \gamma^2 m^2 c^4 \left(1 - \frac{1}{\gamma^2}\right)$$

And:

$$p^2c^2 = \gamma^2 m^2 c^4 - m^2 c^4$$

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Momentum and Energy

$$p^2c^2 = \gamma^2 m^2 c^4 - m^2 c^4$$

The first term on the right-hand side is just E^2, and the second is E_0^2:

$$p^2c^2 = E^2 - E_0^2$$

Rearranging, we obtain a relation between energy and momentum.

$$E^2 = p^2c^2 + E_0^2$$

or:

$$E^2 = p^2c^2 + m^2c^4$$

This equation relates the total energy of a particle with its momentum. The quantities $(E^2 - p^2c^2)$ and m are invariant quantities.

Note that when a particle's velocity is zero and it has no momentum, this equation correctly gives E_0 as the particle's total energy.

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Invariant "Square" Equations

Invariant quantities are equal in all inertial reference frames

- All observers will measure the same value for the rest mass mc^2 and for the space-time interval Δs.

Invariant quantities are called "4-vectors"

- Rest mass mc^2 is given by four quantities: E, p_x, p_y, p_z
- Space-time interval Δs is given by four quantities: t, x, y, z

Rest Mass E_o: $$\left(mc^2\right)^2 = E^2 - \left(pc\right)^2 \quad \text{(Square Eqn.)}$$

Space-Time Interval: $$\left(\Delta s\right)^2 = \left(c\Delta t\right)^2 - \left(\Delta x\right)^2$$

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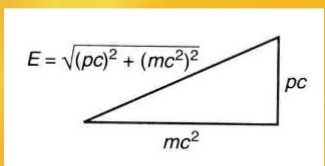
Invariant Equation: Classical vs. Relativistic



Classical Limit

$E = \gamma\, mc^2$

E >> pc gives $E \approx mc^2$ ($\gamma \approx 1$)

→ $K \approx \frac{1}{2} mu^2$ or $p^2/2m$

If $K/mc^2 \approx 1\%$, then K approximation is accurate to ≈1.5%.

Very Relativistic Limit

$E = \gamma\, mc^2$

E >> mc^2 gives $E \approx pc$

Accurate to 1% or better if E > $8mc^2$

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6 Variables given by 5 Eqns. (3 def., simple, square)

Requires mc^2 $\quad \boxed{E} = \gamma mc^2 = K + mc^2 = \sqrt{(pc)^2 + (mc^2)^2}$

Requires mc^2 $\quad \boxed{pc} = \gamma (mc^2)(u/c) = \sqrt{E^2 - (mc^2)^2}$

Requires mc^2 $\quad \boxed{K} = E - mc^2$

$$\Rightarrow \quad \boxed{mc^2} = \frac{E}{\gamma} = \frac{pc}{\gamma(u/c)} = E - K = \sqrt{E^2 - (pc)^2}$$

$$\boxed{\frac{u}{c}} = \frac{pc}{\gamma mc^2} = \sqrt{1 - \frac{1}{\gamma^2}}$$

$$\boxed{\gamma} = \frac{E}{mc^2} = \frac{1}{\sqrt{1-(u/c)^2}}$$

Problem: Find E, p, and K (given u, mc^2)

Find the total energy E, momentum p (MeV/c), and kinetic energy K for an electron with rest mass 0.511 MeV and speed u = 0.5c.

u gives γ $\quad \boxed{\gamma} = \frac{1}{\sqrt{1-(u/c)^2}} = \frac{1}{\sqrt{1-(0.5)^2}} = \boxed{1.155}$

$$\boxed{E} = \gamma \mathrm{mc}^2 = 1.155(0.511\,\mathrm{MeV}) = \boxed{0.59\,\mathrm{MeV}}$$

$$\boxed{p} = \gamma \mathrm{mu} = 1.155(0.511\,\mathrm{MeV}/c^2)(0.5c) = \boxed{0.295\,\mathrm{MeV/c}}$$

$$\boxed{K} = E - mc^2 = 0.59\,\mathrm{MeV} - 0.511\,\mathrm{MeV} = \boxed{0.079\,\mathrm{MeV}}$$

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Problem: Find pc and u (given E, mc^2)

Find the momentum pc (MeV) and speed u of an electron with rest mass 0.511 MeV and total energy E = 10 MeV.

Use Square Eqn.

$$\boxed{pc} = \sqrt{E^2 - (mc^2)^2} = \sqrt{(10\text{ MeV})^2 - (0.511\text{ MeV})^2} = \boxed{9.987\text{ MeV}}$$

$$\boxed{\gamma} = \frac{E}{mc^2} = \frac{10\text{ MeV}}{0.511\text{ MeV}} = \boxed{19.57} \qquad \text{using } E = \gamma mc^2$$

$$\boxed{\frac{u}{c}} = \sqrt{1 - \frac{1}{\gamma^2}} = \sqrt{1 - \frac{1}{(19.57)^2}} = \boxed{0.9987} \qquad \text{using } \gamma = \frac{1}{\sqrt{1-\beta^2}}$$

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Problem: Find mc^2 and u (given p, E)

Find the rest mass and speed u of a particle with momentum pc = 300 MeV and total energy E = 3500 MeV.

Use Square Eqn.

$$\boxed{mc^2} = \sqrt{E^2 - (pc)^2} = \sqrt{(3500\text{ MeV})^2 - (300\text{ MeV})^2} = \boxed{3487\text{ MeV}}$$

$$\boxed{\gamma} = \frac{E}{mc^2} = \frac{3500\text{ MeV}}{3487\text{ MeV}} = \boxed{1.00373} \qquad \text{using } E = \gamma mc^2$$

$$\boxed{\frac{u}{c}} = \sqrt{1 - \frac{1}{\gamma^2}} = \sqrt{1 - \frac{1}{(1.00373)^2}} = \boxed{0.086} \qquad \text{using } \gamma = \frac{1}{\sqrt{1-\beta^2}}$$

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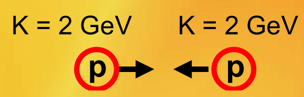
Problem: Find E, pc and u (given K, mc^2)

A 2-GeV proton hits another 2-GeV proton in a head-on collision. Find the total energy E, momentum pc, and velocity u of each proton.



Use Simple AND Square Eqns.

$$\boxed{E} = K + mc^2 = 2\text{ GeV} + 938\text{ MeV} = \boxed{2.938\text{ GeV}}$$

$$\boxed{pc} = \sqrt{E^2 - mc^2} = \sqrt{(2.938\text{ GeV})^2 - (0.938\text{ GeV})^2} = \boxed{2.78\text{ GeV}}$$

$$\boxed{\gamma} = \frac{E}{mc^2} = \frac{2.938\text{ GeV}}{0.938\text{ GeV}} = \boxed{3.13} \quad \text{using } E = \gamma mc^2$$

$$\boxed{\frac{u}{c}} = \sqrt{1 - \frac{1}{\gamma^2}} = \sqrt{1 - \frac{1}{(3.13)^2}} = \boxed{0.948} \quad \text{using } \gamma = \frac{1}{\sqrt{1-\beta^2}}$$

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Problem: Find mc^2 and K

A Σ particle decays into a neutron (pc = 4702 MeV) and pion (pc = 169 MeV). Find the total rest mass and kinetic energy of the Σ particle.

$$\boxed{E_n} = \sqrt{(pc)_n^{\ 2} + \left(mc^2\right)_n^{\ 2}} = \sqrt{(4702\text{ MeV})^2 + (940\text{ MeV})^2} = \boxed{4795\text{ MeV}}$$

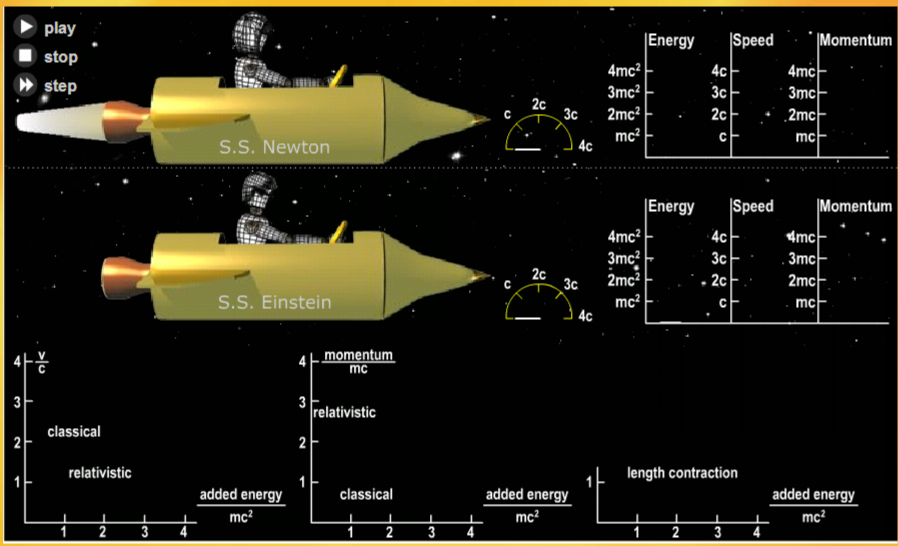
$$\boxed{E_\pi} = \sqrt{(pc)_\pi^{\ 2} + \left(mc^2\right)_\pi^{\ 2}} = \sqrt{(169\text{ MeV})^2 + (140\text{ MeV})^2} = \boxed{219\text{ MeV}}$$

$$\boxed{E_\Sigma} = E_n + E_\pi = 4795\text{ MeV} + 219\text{ MeV} = \boxed{5014\text{ MeV}}$$

$$\boxed{\left(mc^2\right)_\Sigma} = \sqrt{E_\Sigma^{\ 2} - (pc_\Sigma)^2} = \sqrt{(5014\text{ MeV})^2 - (4871\text{ MeV})^2} = \boxed{1189\text{ MeV}}$$

$$\text{where } pc_\Sigma = pc_n + pc_\pi = 4702\text{ MeV} + 169\text{ MeV} = 4871\text{ MeV}$$

$$\boxed{K_\Sigma} = E_\Sigma - \left(mc^2\right)_\Sigma = 5014\text{ MeV} - 1189\text{ MeV} = \boxed{3825\text{ MeV}}$$



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Computations in Modern Physics

We were taught in introductory physics that the international system of units is preferable when doing calculations in science and engineering.

In modern physics, a somewhat different, more convenient set of units is often used.

The smallness of quantities often used in modern physics suggests some practical changes.

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The Electron Volt (eV)

Artist's rendition of an electron (don't take this too seriously)

The work done in accelerating a charge through a potential difference is given by $W = qV$. For a proton, with the charge $e = 1.602 \times 10^{-19}$ C and a potential difference of 1 V, the work done is:

$$W = (1.602 \times 10^{-19}\ \text{C})(1\ \text{V}) = 1.602 \times 10^{-19}\ \text{J}$$

The work done to accelerate the proton across a potential difference of 1 V could also be written as:

$$W = (1\ e)(1\ \text{V}) = 1\ \text{eV}$$

Thus eV, pronounced "electron volt," is also a unit of energy. It's related to the SI (*Système International*) unit joule by:

$$1\ \text{eV} = 1.602 \times 10^{-19}\ \text{J}$$

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Rest Energy

Rest energy of a particle ($E_0 = mc^2$):

Example: E_0 (proton)

$$E_0(\text{proton}) = (1.67 \times 10^{-27}\ \text{kg})(3 \times 10^8\ \text{m/s})^2 = 1.50 \times 10^{-10}\ \text{J}$$

Atomic mass unit (amu) (the number of nucleons in the nucleus):

Example: carbon-12

$$= 1.50 \times 10^{-10}\ \text{J}\ \frac{1\ \text{eV}}{1.602 \times 10^{-19}\ \text{J}} = 9.38 \times 10^8\ \text{eV}$$

Mass (^{12}C atom) $= \dfrac{12\ \text{g/mol}}{6.02 \times 10^{23}\ \text{atoms/mol}}$

$= 1.99 \times 10^{-23}\ \text{g/atom}$

Mass (^{12}C atom) $= 1.99 \times 10^{-26}\ \text{kg} = 12\ \text{u/atom}$

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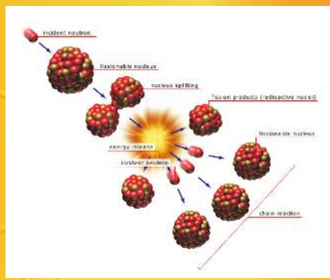
Binding Energy



The equivalence of mass and energy becomes apparent when we study the binding energy of systems like atoms and nuclei that are formed from individual particles.

The potential energy associated with the force keeping the system together is called the **binding energy** E_B.

The binding energy is *the difference between the rest energy of the individual particles and the rest energy of the combined bound system*.

$$M_{\text{bound system}} c^2 + E_B = \sum_i m_i c^2$$

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