King Saud University

College of Sciences

Department of Mathematics

151 Math Exercises

(3,3)

Methods of Proof

"Mathematical Induction"

(STRONG INDUCTION)

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Strong Induction

STRONG INDUCTION To prove that $P(n)$ is true for all positive integers n, where $P(n)$ is a propositional function, we complete two steps:

BASIS STEP: We verify that the proposition $P(1)$ is true.

INDUCTIVE STEP: We show that the conditional statement

$[P(1) \wedge P(2) \wedge \cdot \cdot \cdot \wedge P(k)] \rightarrow P(k+1)$ is true for all positive integers k.

Exercises

1. Assume $\{a_n\}_{n=1}^{\infty}$ is a sequence defined as:

$$a_1 = 3\ , \quad a_2 = 6\ \ , \ a_n = a_{n-1} + a_{n-2} \ :\forall n \geq 3$$

Prove that $3|\, a_n$ for all positive integers n .

Solution: Let $P(n)$ be the proposition , $P(n):\ 3|\, a_n\ ,\ \forall\, n \geq 1 \Rightarrow a_n = 3c : c \in \mathbb{N}$

BASIS STEP: $P(1):\ 3|\ a_1 = 3 \ \Rightarrow\ \therefore\ P(1)\ \ is\ true$.

$P(2):\ 3|\ a_2 = 6 \ \Rightarrow \therefore\ P(2)\ is\ true$.

INDUCTIVE STEP: We assume that $P(1), P(2), \ldots, P(k-2), P(k-1)\ , P(k)$ (*)

All are hold for integer $k \geq 2$.

Under this assumption, it must be shown that $P(k+1)$ is also true .

$$P(\,k+1\,):\qquad a_{k+1} = a_k + a_{k-1} \qquad\qquad (**)$$

$\because\ a_{k-1}$ و a_k both are true, (from inductive hypothesis *) $\Rightarrow$

$P(k):\ 3|\, a_k \quad \Rightarrow a_k = 3c_1 : c_1 \in \mathbb{N}$

$P(k-1):\ 3|\, a_{k-1} \quad \Rightarrow a_{k-1} = 3c_2 \quad : \ c_2 \in \mathbb{N}$

$\xRightarrow{by\ subist.into(**)}$

$$a_{k+1} = a_k + a_{k-1} = 3c_1 + 3c_2 = 3(c_1 + c_2) = 3c$$

$$: c = (c_1 + c_2) \in \mathbb{N}$$

$\therefore\ a_{k+1} = 3c \ \Rightarrow\ 3|\, a_{k+1} \ \Rightarrow \therefore\ \ P(\,k+1\,)$ is true .

Then $P(n\,)$ is true for $\forall\, n \geq 1$. #

2. Assume $\{a_n\}_{n=1}^{\infty}$ is a sequence defined as:

$$a_1 = 8\,,\ a_2 = 4\ ,\ \boxed{a_n = a_{n-1} + a_{n-2}\ :\forall n \geq 3}$$

Prove that a_n is even for all positive integers n .

Solution:

3. Assume $\{a_n\}_{n=0}^{\infty}$ is a sequence defined as:

$$a_0 = 9\,,\ a_1 = 15\ ,\ \ a_n = \frac{a_{n-1}\,a_{n-2}}{3} + 6 \quad : \forall n \geq 2$$

Prove that $3|a_n$ for all nonnegative integers *n*.

Solution:

4. Assume $\{b_n\}_{n=1}^{\infty}$ is a sequence defined as:

$$b_1 = 1 , \ b_2 = 2 \ , \ b_n = 2b_{n-1} - \ b_{n-2} \quad : \forall n \geq 3$$

Prove that $b_n = n$ for all positive integers n.

Solution:

5. Assume $\{a_n\}_{n=1}^{\infty}$ is a sequence defined as:

$$a_1 = -1\ ,\ \ a_2 = -\frac{1}{2}\ ,\ a_3 = -\sqrt{10}\ \ ,\ \ a_n = a_{n-1}.a_{n-2}.a_{n-3}\ \ :\forall n \geq 4$$

Prove that $a_n \leq 0$ for all positive integers *n*.

Solution:

6. Assume $\{a_n\}_{n=1}^{\infty}$ is a sequence defined as:

$$a_1 = 1\,,\ a_2 = 5\ ,\quad a_{n+1} = 2a_n + 3\,a_{n-1}\quad : \forall n \ge 2 \quad (*)$$

Prove that $3^n \le a_{n+1} \le 2\,.\,3^n$ for all positive integers n.

Solution: Let $P(n)$ be the proposition , $P(n):\ 3^n \le a_{n+1} \le 2\,.\,3^n\ ,\ \forall\, n \ge 1$

BASIS STEP: $P(1):\ 3^1 \le a_2 = 5 \le 2\,.\,3^1 = 6 \ \Rightarrow\ \therefore\ P(1)$ *is true* .

INDUCTIVE STEP: We assume that $P(1), P(2), \dots, P(k-2), P(k-1)\,, P(k)$ (**)
All are hold for integer $k \ge 1$.

Under this assumption, it must be shown that $P(k+1)$ is also true .

from $(*) \Rightarrow \quad P(k+1):\quad a_{k+2} = 2\,a_{k+1} + 3\,a_k \qquad (***)$

$\because\ a_{k+1}$ و a_k both are true, (from inductive hypothesis **) $\Rightarrow$

$P(k):\ 3^k \le a_{k+1} \le 2\,.\,3^k \Rightarrow 2.\,3^k \le 2a_{k+1} \le 4\,.\,3^k \qquad (1)$

$P(k-1):\ 3^{k-1} \le a_k \le 2\,.\,3^{k-1} \ \Rightarrow 3.\,3^{k-1} \le 3a_k \le 2\,.3.\,3^{k-1}$

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$$3^k \le 3a_k \le 2\,.\,3^k \qquad (2)$$

$(1) + (2) \quad \Rightarrow \quad 2.\,3^k + 3^k \le 2a_{k+1} + 3a_k \le 4\,.\,3^k + 2\,.\,3^k$

$\xRightarrow{\text{by subist.into}(***)}$

$$3\ 3^k \le a_{k+2} \le 6\ 3^k$$

$$3^{k+1} \le a_{k+2} \le 2.3.\,3^k$$

$$3^{k+1} \le a_{k+2} \le 2\ 3^{k+1}$$

$\Rightarrow \qquad \therefore\ P(k+1)$ is true

$\therefore\ P(n)$ is true for $\forall\, n \ge 1$

#

7. Assume $\{a_n\}_{\mathrm{n=0}}^{\infty}$ is a sequence defined as:

$$a_0 = 1 \ , \quad a_n = 2a_{n-1} + 1 \quad : \forall n \geq 1$$

Prove that $a_n = 2^{n+1} - 1$ for all nonnegative integers *n*.

Solution:

8. Assume $\{a_n\}_{n=1}^{\infty}$ is a sequence defined as:

$$a_1 = 3\,,\ a_2 = 9\,,\ a_3 = 15\ ,\ a_n = a_{n-1} + a_{n-2} + a_{n-3}\ :\forall n \geq 4$$

Prove that 3 divides a_n for all positive integers n.

Solution:

9. Assume $\{a_n\}_{n=0}^{\infty}$ is a sequence defined as:

$$a_0 = 1\ ,\ a_1 = 1\ ,\ a_n = 2a_{n-1} + a_{n-2}\ :\forall n \geq 2$$

Prove that a_n is odd for all nonnegative integers *n*.

Solution:

10. Assume $\{a_n\}_{n=0}^{\infty}$ is a sequence defined as:

$$a_0 = 0\,,\ a_1 = 4\ ,\ \ a_n = -2a_{n-1} + 3\,a_{n-2}\ \ : \forall n \geq 2 \qquad (*)$$

Prove that $a_n = 1 - (-3)^n$ for all nonnegative integers *n*.

Solution:

11. Assume $\{a_n\}_{n=0}^{\infty}$ is a sequence defined as:

$$a_0 = 1 \ , \quad a_1 = 2 \ , \ a_n = 4a_{n-1} - 3a_{n-2} \ : \forall n \geq 2$$

Prove that $a_n = 3^n - 1$ for all nonnegative integers *n*.

Solution:

12. Assume $\{a_n\}_{n=0}^{\infty}$ is a sequence defined as:

$$a_0 = 9\,,\ a_1 = 15\ ,\ a_2 = 3 \qquad a_n = \frac{a_{n-1}\,a_{n-2}\,a_{n-3}}{9} + 6 \quad : \forall n \geq 3$$

Prove that $3|a_n$ for all nonnegative integers *n*.

Solution:

13. Assume $\{u_n\}_{n=1}^{\infty}$ is a sequence defined as:

$$u_1 = 1\ ,\ u_2 = 2\ , u_3 = 3\ \ ,\ \ u_n = 3u_{n-1} - u_{n-2} - u_{n-3}\ : \forall n \geq 4$$

Prove that $u_n = n$ for all positive integers n.

Solution:

14. Assume $\{a_n\}_{n=1}^{\infty}$ is a sequence defined as:

$$a_1 = 1\,,\ a_2 = 2\,, a_3 = 3\ ,\ \ a_n = \frac{a_{n-1} + a_{n-2} + a_{n-3}}{3}\ \ : \forall n \geq 4$$

Prove that $1 \leq a_n \leq 3$ for all positive integers *n*.

Solution:

15. Assume $\{u_n\}_{n=1}^{\infty}$ is a sequence defined as:

$$u_1 = \frac{3}{4} \; , \; u_2 = \frac{8}{13} \quad , \quad u_n = \frac{3\,u_{n-1} + 2\,u_{n-2}}{3} \quad : \forall n \geq 3$$

Prove that $u_n < 1$ for all positive integers *n*.

Solution:

16. Assume $\{u_n\}_{n=1}^{\infty}$ is a sequence defined as:

$$u_1 = 2\,,\ u_2 = 4 \quad , \quad u_n = \frac{2u_{n-1} + u_{n-2} + 8}{3} \quad : \forall n \geq 3$$

Prove that $u_n = 2n$ for all positive integers *n*.

Solution:

17. Assume $\{a_n\}_{n=0}^{\infty}$ is a sequence defined as:

$$a_0 = 1 , \quad a_1 = 2 , \; a_2 = 3 , \quad a_n = a_{n-1} + a_{n-2} + 2a_{n-3} \; : \forall n \geq 3$$

Prove that $a_n \leq 3^n$ for all nonnegative integers *n*.

Solution:

18. Assume $\{a_n\}_{\mathrm{n}=0}^{\infty}$ is a sequence defined as:

$$a_0 = 1\ , \quad a_1 = 2 \quad , \quad a_n = 4a_{n-1} - 4a_{n-2} \quad : \forall n \geq 2$$

Prove that $a_{\mathrm{n}} = 2^n$ for all nonnegative integers *n*.

Solution:

19. Assume $\{a_n\}_{n=0}^{\infty}$ is a sequence defined as:

$$a_0 = 1 , \quad a_1 = 1 \quad , \quad a_n = 4a_{n-1} - 4a_{n-2} \quad : \forall n \geq 2$$

Prove that $a_n = 2^n - n2^{n-1}$ for all nonnegative integers *n*.

Solution:

20. Assume $\{a_n\}_{n=1}^{\infty}$ is a sequence defined as:

$$a_1 = 1\,,\ a_2 = 3 \quad , \quad a_n = a_{n-1} + a_{n-2} \quad : \forall n \geq 3$$

Prove that $a_n = (\frac{1+\sqrt{5}}{2})^n + (\frac{1-\sqrt{5}}{2})^n$ for all positive integers *n*.

Solution:

21. Assume $\{a_n\}_{n=1}^{\infty}$ is a "Fibonacci" sequence defined as:

$$a_1 = 1\,,\ a_2 = 1 \quad , \quad a_n = a_{n-1} + a_{n-2} \quad : \forall n \geq 3$$

Prove that $a_n \leq (\frac{1+\sqrt{5}}{2})^n$ for all positive integers *n*.

Solution:

22. Assume $\{a_n\}_{n=0}^{\infty}$ is a sequence defined as:

$$a_0 = 1\,,\ a_1 = 1\ ,a_2 = 3\ ,\ \ a_n = a_{n-1} + a_{n-2}\ + a_{n-3}\ \ :\forall n \geq 2$$

Prove that $a_n < 3^n$ for all nonnegative integers *n*.

Solution:

23. Assume $\{a_n\}_{\mathrm{n}=0}^{\infty}$ is a sequence defined as:

$$a_0 = 2\ ,\ a_1 = 4\ , a_2 = 6\ ,\ a_n = 5a_{n-3} \quad : \forall n \geq 3$$

Prove that $2|a_{\mathrm{n}}$ for all nonnegative integers *n*.

Solution:

24. Assume $\{a_n\}_{n=1}^{\infty}$ is a "Fibonacci" sequence defined as:

$$a_1 = 1 , \ a_2 = 2 \quad , \quad a_n = 2a_{n-1} + a_{n-2} \quad : \forall n \geq 2$$

Prove that $a_n \leq (\frac{5}{2})^n$ for all positive integers *n*.

Solution: