Chapter 4: Parameter Estimation

- *Method of moments:*

$AR(1): y_t = \varepsilon_t + \phi_1 y_{t-1}$

$$\hat{\phi}_1 = \hat{\rho}_1 = r_1$$

$AR(2): y_t = \varepsilon_t + \phi_1 y_{t-1} + \phi_2 y_{t-2}$

$$\hat{\phi}_1 = \frac{r_1(1-r_2)}{1-r_1^2} \qquad \hat{\phi}_2 = \frac{r_2-r_1^2}{1-r_1^2}$$

$MA(1): y_t = \varepsilon_t - \theta_1 \varepsilon_{t-1}$

$$r_1 = \frac{-\theta_1}{1+\theta_1^2}$$

$$r_1\, \theta^2 + \theta + r_1 = 0$$

$$\hat{\theta}_1 = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-1 \pm \sqrt{1-4r_1^2}}{2r_1}$$

ACF of MA (1)

$$\rho(k) = \begin{cases} 1 & k = 0 \\ -\dfrac{\theta_1}{(1+\theta_1^2)} & k = 1 \\ 0 & k \geq 2 \end{cases}$$

Estimation the σ_α^2

for $AR(p): \hat{\sigma}_\varepsilon^2 = \left(1 - \hat{\phi}_1 r_1 - \hat{\phi}_2 r_2 - \cdots - \hat{\phi}_p r_p\right) S^2$

for $AR(1): \hat{\sigma}_\varepsilon^2 = (1 - r_1^2) S^2 \quad ; \quad \hat{\phi}_1 = r_1$

for $AR(2): \hat{\sigma}_\varepsilon^2 = \left(1 - \hat{\phi}_1 r_1 - \hat{\phi}_2 r_2\right) S^2 \quad ; \quad \hat{\phi}_1 = \frac{r_1(1-r_2)}{1-r_1^2} \quad and \quad \hat{\phi}_2 = \frac{r_2-r_1^2}{1-r_1^2}$

for $MA(q): \hat{\sigma}_\varepsilon^2 = \frac{S^2}{(1+\hat{\theta}_1^2+\hat{\theta}_2^2+\cdots+\hat{\theta}_q^2)}$

for $MA(1): \hat{\sigma}_\varepsilon^2 = \frac{S^2}{(1+\hat{\theta}_1^2)}$

for $ARMA(1,1): \hat{\sigma}_\varepsilon^2 = \frac{1-\hat{\phi}_1^2}{(1-2\,\hat{\theta}_1\hat{\phi}_1+\hat{\theta}_1^2)} S^2$

- *Least squares methods:*

for $AR(1)$:

$$(y_t - \mu) = \varepsilon_t + \phi_1(y_{t-1} - \mu)$$
$$\varepsilon_t = (y_t - \mu) - \phi_1(y_{t-1} - \mu)$$
$$S(\phi_1 ,\ \mu) = \sum_{t=2}^{n} \varepsilon_t^2 = \sum_{t=2}^{n}[(y_t - \mu) - \phi_1(y_{t-1} - \mu)]^2$$

$$\frac{\partial S}{\partial \mu} = 2\sum_{t=2}^{n}[(y_t - \mu) - \phi_1(y_{t-1} - \mu)][-1 + \phi_1] = 0$$
$$\Rightarrow \sum_{t=2}^{n}[(y_t - \mu) - \phi_1(y_{t-1} - \mu)] = 0$$
$$\Rightarrow \sum_{t=2}^{n}[y_t - \mu - \phi_1 y_{t-1} + \phi_1\mu] = 0$$
$$\Rightarrow \sum_{t=2}^{n} y_t - \sum_{t=2}^{n} \mu - \phi_1 \sum_{t=2}^{n} y_{t-1} + \sum_{t=2}^{n} \phi_1\mu = 0$$
$$\Rightarrow \sum_{t=2}^{n} y_t - (n-1)\mu - \phi_1 \sum_{t=2}^{n} y_{t-1} + (n-1)\phi_1\mu = 0$$
$$\Rightarrow \sum_{t=2}^{n} y_t - \phi_1 \sum_{t=2}^{n} y_{t-1} - (n-1)\mu + (n-1)\phi_1\mu = 0$$
$$\Rightarrow \sum_{t=2}^{n} y_t - \phi_1 \sum_{t=2}^{n} y_{t-1} - (1-\phi_1)(n-1)\,\mu = 0$$
$$\Rightarrow \sum_{t=2}^{n} y_t - \phi_1 \sum_{t=2}^{n} y_{t-1} = (1-\phi_1)(n-1)\,\mu$$

$$\Rightarrow \boxed{\hat{\mu} = \frac{\sum_{t=2}^{n} y_t - \phi_1 \sum_{t=2}^{n} y_{t-1}}{(n-1)(1-\phi_1)}}$$

for large n

$$\sum_{t=2}^{n} \frac{y_t}{n-1} = \sum_{t=2}^{n} \frac{y_{t-1}}{n-1} \approx \bar{y}$$

$$\frac{\partial S}{\partial \phi_1} = 2\sum_{t=2}^{n}[(y_t - \bar{y}) - \phi_1(y_{t-1} - \bar{y})](y_{t-1} - \bar{y}) = 0$$
$$\Rightarrow \sum_{t=2}^{n}(y_t - \bar{y})(y_{t-1} - \bar{y}) - \phi_1 \sum_{t=2}^{n}(y_{t-1} - \bar{y})^2 = 0$$

$$\Rightarrow \boxed{\hat{\phi}_1 = \frac{\sum_{t=2}^{n}(y_t - \bar{y})(y_{t-1} - \bar{y})}{\sum_{t=2}^{n}(y_{t-1} - \bar{y})^2}}$$

$MA(1)$ *see page 95-97*

Question 1:

Using the method of moments, answer the following questions:

a. *What is the estimate of the variance of the process* $\{y_t\}$, *i.e.* $\gamma(0)$ *in the ARMA(p,q) models?*

$\gamma(0)$ *is estimated by the method of moments as* $S^2 = \sum_{i=1}^{n} \frac{(y_i - \bar{y})^2}{n-1}$

b. *For the model* $y_t = \varepsilon_t - \theta_1 \varepsilon_{t-1}$:

i. *obtain the estimate of the parameter* θ_1.

$$r_1 = \frac{-\theta_1}{1+\theta_1^2}$$

$$r_1 \theta^2 + \theta + r_1 = 0$$

$$\Longrightarrow \hat{\theta}_1 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-1 \pm \sqrt{1 - 4r_1^2}}{2r_1}$$

$$\rho(k) = \begin{cases} 1 & k = 0 \\ -\dfrac{\theta_1}{(1+\theta_1^2)} & k = 1 \\ 0 & k \geq 2 \end{cases}$$

ii. *obtain the estimate of white noise variance* $\hat{\sigma}_\varepsilon^2$.

$$\gamma_0 = Var(y_t) = Var(\varepsilon_t - \theta_1 \varepsilon_{t-1}) = Var(\varepsilon_t) + Var(\theta_1 \varepsilon_{t-1})$$

$$\gamma_0 = \sigma_\varepsilon^2 + \theta_1^2 \sigma_\varepsilon^2$$

$$\gamma_0 = \sigma_\varepsilon^2 (1 + \theta_1^2)$$

$$\boxed{\hat{\sigma}_\varepsilon^2 = \frac{S^2}{(1+\hat{\theta}_1^2)}}$$

Question 2:

Using the method of moments, answer the following questions:

a. *For the model* $y_t = \phi_1 y_{t-1} + \varepsilon_t$:

We have the Yule-Walker equation

$$\rho_k = \phi_1 \rho_{k-1} \ ; \ k = 1,2,3,\ldots$$

replace ρ_k *by* r_k *we got:*

$$r_k = \hat{\theta}_1 r_{k-1} \quad ; \ k = 1,2,3,\ldots$$

for $k = 1$: $\quad r_1 = \hat{\theta}_1 r_0 = \hat{\theta}_1$

$$\boxed{\hat{\theta}_1 = r_1}$$

b. *Obtain the estimate of white noise variance* $\hat{\sigma}_\varepsilon^2$

for AR (1), we have

$$\gamma_0 = Var(y_t) = Var(\phi_1 y_{t-1} + \varepsilon_t)$$

$$\gamma_0 = Var(\phi_1 y_{t-1}) + Var(\varepsilon_t)$$

$$\gamma_0 = \phi_1 \gamma_1 + \sigma_\varepsilon^2$$

$$\gamma_0 = \phi_1 \rho_1 \gamma_0 + \sigma_\varepsilon^2 \qquad \text{where, } \rho_1 = \frac{\gamma_1}{\gamma_0} \Longrightarrow \boxed{\gamma_1 = \rho_1 \gamma_0}$$

$$\hat{\sigma}_\varepsilon^2 = \hat{\gamma}_0 - \hat{\phi}_1 r_1 \hat{\gamma}_0$$

$$\hat{\sigma}_\varepsilon^2 = \hat{\gamma}_0 \left(1 - \hat{\phi}_1 r_1\right)$$

$$\boxed{\hat{\sigma}_\varepsilon^2 = S^2 \left(1 - r_1^2\right)} \qquad \text{where, } \hat{\phi}_1 = r_1$$

Question 3:

Assume that 100 observations from an ARMA (1,1) model

$$Z_t - \phi_1 Z_{t-1} = \alpha_t - \theta_1 \alpha_{t-1}$$

Gave the following estimates $\hat{\sigma}_\varepsilon^2 = 10$, $\hat{\rho}_1 = 0.523$ *and* $\hat{\rho}_2 = 0.418$.

Find initial estimates for ϕ_1, θ_1 *and* σ_ε^2.

The ACF of ARMA (1,1) is

$$\rho_k = \begin{cases} \dfrac{(\phi_1 - \theta_1)(1 - \phi_1\theta_1)}{(1 + \theta_1^2 - 2\phi_1\theta_1)} & ; k = 1 \\ \phi_1 \rho_{k-1} & ; k = 2,3,\dots \end{cases}$$

- *Estimating* ϕ_1:

$\rho_2 = \phi_1 \rho_1$

$r_2 = \hat{\phi}_1 r_1$

$$\hat{\phi}_1 = \frac{r_2}{r_1} = \frac{0.418}{0.523} \implies \boxed{\hat{\phi}_1 = 0.799}$$

- *Estimating* θ_1:

$$r_1 = \frac{(\hat{\phi}_1 - \hat{\theta}_1)(1 - \hat{\phi}_1\hat{\theta}_1)}{1 + \hat{\theta}_1^2 - 2\hat{\phi}_1\hat{\theta}_1} = \frac{\hat{\phi}_1 - \hat{\phi}_1^2\hat{\theta}_1 - \hat{\theta}_1 + \hat{\phi}_1\hat{\theta}_1^2}{1 + \hat{\theta}_1^2 - 2\hat{\phi}_1\hat{\theta}_1}$$

$$0.523 = \frac{0.799 - 0.638\,\hat{\theta}_1 - \hat{\theta}_1 + 0.799\,\hat{\theta}_1^2}{1 + \hat{\theta}_1^2 - 1.598\,\hat{\theta}_1}$$

$$0.523 = \frac{0.799 - 1.638\,\hat{\theta}_1 + 0.799\,\hat{\theta}_1^2}{1 + \hat{\theta}_1^2 - 1.598\,\hat{\theta}_1}$$

$$0.523\left(1+\hat{\theta}_1^2-1.598\,\hat{\theta}_1\right)=0.799-1.638\,\hat{\theta}_1+0.799\,\hat{\theta}_1^2$$

$$0.523+0.523\,\hat{\theta}_1^2-0.836\,\hat{\theta}_1=0.799-1.638\,\hat{\theta}_1+0.799\,\hat{\theta}_1^2$$

$$0.799-1.638\,\hat{\theta}_1+0.799\,\hat{\theta}_1^2-\left(0.523+0.523\,\hat{\theta}_1^2-0.836\,\hat{\theta}_1\right)=0$$

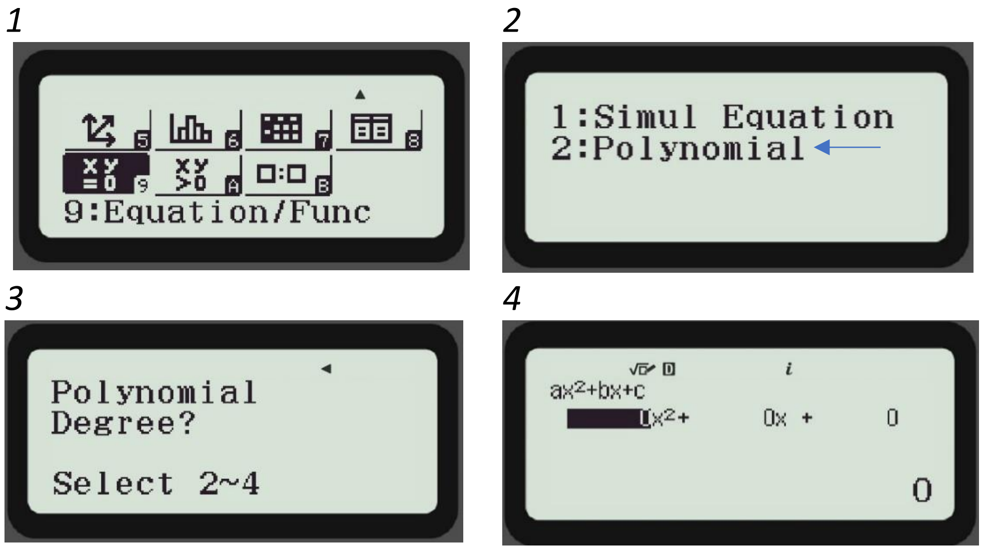
$$0.276\,\hat{\theta}_1^2-0.802\,\hat{\theta}_1+0.276=0$$

$$\hat{\theta}_1=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$\hat{\theta}_1=\frac{-0.802\pm\sqrt{0.802^2-4(0.276)(0.276)}}{2(0.276)}$$

$$\hat{\theta}_1=2.507\quad or\quad \hat{\theta}_1=0.399$$

or we can use the calculator



We will choose 0.399 to satisfy invertibility condition, which is $|\theta_1|<1$.

- *Estimating* σ_ε^2:

$$\sigma_\varepsilon^2=\frac{1-\hat{\phi}_1^2}{1-2\hat{\theta}_1\hat{\phi}_1+\theta_1^2}S^2$$

$$\sigma_\varepsilon^2=\frac{1-0.799^2}{1-2(0.799)(0.399)+0.399^2}\;(10)=6.932$$

Question 4:

Assume that 100 observations from an AR(2) model

$$Z_t = \phi_1 Z_{t-1} + \phi_2 Z_{t-2} + \alpha_t$$

Gave the following sample ACF: $\hat{\rho}_1 = 0.8,\ \hat{\rho}_2 = 0.5\ \ and\ \hat{\rho}_3 = 0.4$

Estimate ϕ_1 *and* ϕ_2.

The auto-correlation function for AR(2) is : $\rho_k = \phi_1(\rho_{k-1}) + \phi_2(\rho_{k-2}) \quad ; k = 1,2, \ldots$

$\hat{\rho}_k = r_k$

$$r_1 = \hat{\phi}_1 r_0 + \hat{\phi}_2 r_1$$
$$r_1 = \hat{\phi}_1(1) + \hat{\phi}_2 r_1$$
$$\Longrightarrow \hat{\phi}_1 = r_1 - \hat{\phi}_2 r_1 \quad \ldots \quad (1)$$

$$r_2 = \hat{\phi}_1 r_1 + \hat{\phi}_2 r_0$$
$$r_2 = \hat{\phi}_1 r_1 + \hat{\phi}_2$$
$$\Longrightarrow \hat{\phi}_2 = r_2 - \hat{\phi}_1 r_1 \quad \ldots \quad (2)$$

substitute (2) in (1):	*substitute (1) in (2):*
$\hat{\phi}_1 = r_1 - \hat{\phi}_2 r_1$ $\hat{\phi}_1 = r_1(1 - \hat{\phi}_2)$ $\hat{\phi}_1 = r_1\left(1 - (r_2 - \hat{\phi}_1 r_1)\right)$ $\hat{\phi}_1 = r_1(1 - r_2 + \hat{\phi}_1 r_1)$ $\hat{\phi}_1 = r_1 - r_1 r_2 + \hat{\phi}_1 r_1^2$ $\hat{\phi}_1 - \hat{\phi}_1 r_1^2 = r_1 - r_1 r_2$ $\hat{\phi}_1(1 - r_1^2) = r_1(1 - r_2)$ $\hat{\phi}_1 = \frac{r_1(1-r_2)}{1-r_2^2}$ $\hat{\phi}_1 = \frac{r_1(1-r_2)}{1-r_2^2} = \frac{0.8(1-0.5)}{1-0.8^2} = 1.11$	$\hat{\phi}_2 = r_2 - \hat{\phi}_1 r_1$ $\hat{\phi}_2 = r_2 - (r_1 - \hat{\phi}_2 r_1) r_1$ $\hat{\phi}_2 = r_2 - r_1^2 - \hat{\phi}_2 r_1^2$ $\hat{\phi}_2 + \hat{\phi}_2 r_1^2 = r_2 - r_1^2$ $\hat{\phi}_2(1 + r_1^2) = r_2 - r_1^2$ $\hat{\phi}_2 = \frac{r_2 - r_1^2}{(1+r_1^2)}$ $\hat{\phi}_2 = \frac{r_2 - r_1^2}{1-r_1^2} = \frac{0.5-0.8^2}{1-0.8^2} = -0.3889$

Other solution:

$r_1 = \hat{\phi}_1 + \hat{\phi}_2 r_1$ $0.8 = \hat{\phi}_1 + (0.8)\,\hat{\phi}_2$ $\hat{\phi}_1 = 0.8 - 0.8\,\hat{\phi}_2 \ldots (1)$	$r_2 = \hat{\phi}_1 r_1 + \hat{\phi}_2$ $0.5 = 0.8\,\hat{\phi}_1 + \hat{\phi}_2 \ldots (2)$

Replace $\hat{\phi}_1$ in (2) using (1):

$$0.5 = 0.8\,\hat{\phi}_1 + \hat{\phi}_2$$

$$0.5 = 0.8\left(0.8 - 0.8\,\hat{\phi}_2\right) + \hat{\phi}_2$$

$$0.5 = 0.64 - 0.64\,\hat{\phi}_2 + \hat{\phi}_2$$

$$0.5 = 0.64 + 0.36\,\hat{\phi}_2$$

$$-0.14 = 0.36\,\hat{\phi}_2$$

$$\hat{\phi}_2 = -0.3889$$

from (1)

$$\hat{\phi}_1 = 0.8 - 0.8\,\hat{\phi}_2$$

$$\hat{\phi}_1 = 0.8 - 0.8\,(-0.3889)$$

$$\hat{\phi}_1 = 1.1111$$