Chapter 2: Basic concepts

Question 1:

1- *What is the difference between strict and weak stationarity? When we can say that weak stationarity leads to strict stationarity?*

Strict stationarity:

If the set $(t_1, t_2, \ldots, t_m)$ *is subset of the time units, where* $m = 1,2, \ldots$ *and if* $k = \pm 1, \pm 2, \ldots$, *then the stochastic process* $\{Y_t\}$ *is strictly stationary if the joint probability distribution function of the variables* $\{Y_{t_1}, Y_{t_2}, \ldots, Y_{t_m}\}$ *is the same as the joint probability distribution function of the variables* $\{Y_{t_1+k}, Y_{t_2+k}, \ldots, Y_{t_m+k}\}$ *for any set of time points* $(t_1, t_2, \ldots, t_m)$ *and for any time lag k.*

Weak stationarity:

the stochastic process $\{Y_t\}$ *is weakly stationary if the second order moments exist, and satisfy:*

a) *The mean of the process* μ_t *is constant and do not depend on time t, that is:*

$$\mu_t = E(Y_t) = \mu \;\; ; t = 0, \pm 1, \pm 2, \ldots$$

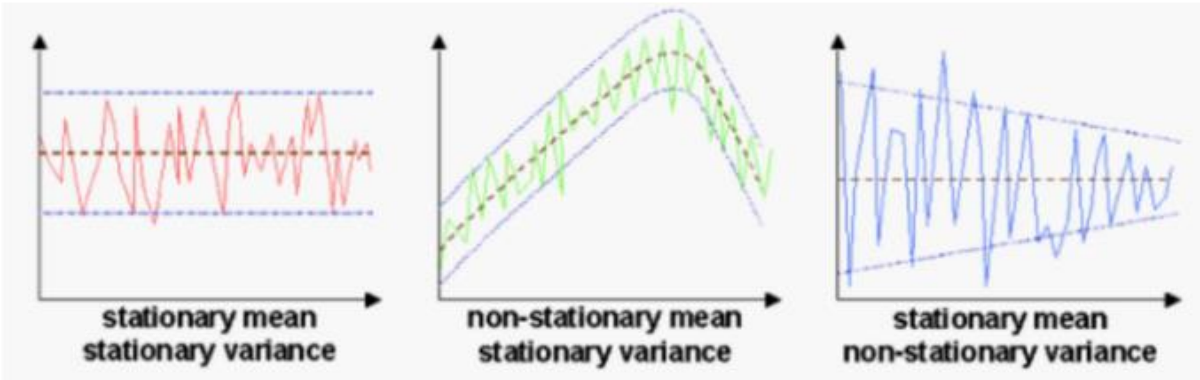
b) *The variance of the process* σ_t^2 *is constant and do not depend on time t, that is:*

$$\sigma_t^2 = V(Y_t) = \gamma(0) \;\; ; t = 0, \pm 1, \pm 2, \ldots$$

c) *The covariance between any two variables depends only on the time lag between them. That is:*

$$Cov(Y_t, Y_{t-k}) = \gamma(k)$$

And since the normal distribution is completely defined through its second order moments, so if the joint distribution of the variables $\{Y_{t_1}, Y_{t_2}, \dots, Y_{t_m}\}$ is the multivariate normal distribution, then the weak stationarity leads to the strict stationarity.



2- *An analyst has a time series data representing number of daily car accidents in a major road at Riyadh city. He applied the techniques of regression analysis to analyze the data set, by considering the dependent variable y_t as the number of daily car accidents, and the independent variable the time indices t=1,2,3,4,… representing days. So he applied the following simple linear regression model, $y_t = \beta_0 + \beta_1 t + \varepsilon_t$ comment on what he have done, do you think his analysis is always valid, discuss.*

Solution:

Since the data set he has is a time series, then one would expect that there exists a serial correlation between the observations. Thus, I think what the analyst has done might be risky without checking that there is no serial correlation in the data. The results he would obtain – if there were actually correlation exist- would not be accurate, and the estimates of the coefficients β_i although unbiased but they do not have the minimum variance property, also the estimates of the standard errors of these coefficients underestimate the true standard errors of the estimated regression coefficient, thus the hypothesis tests and the confidence intervals for these coefficients that uses the t and F distributions are not applicable.

Question 2:

1- Assume the model:

$$Y_t = 1 + \varepsilon_t + \varepsilon_{t-1}$$

where $\{\varepsilon_t\}$ is a sequence of independent and identically distributed random variables with mean zero, and variance σ_ε^2 .
Find ***the autocorrelation function (ACF)*** *for the process $\{Y_t\}$, plot it and comment on the graph.*

Solution:

$$\boxed{ACF = \rho_k = \frac{\gamma_k}{\gamma_0} = \frac{Cov(y_t, y_{t-k})}{Var(y_t)}}$$

- $E(y_t) = E(1 + \varepsilon_t + \varepsilon_{t-1})$
  $= E(1) + E(\varepsilon_t) + E(\varepsilon_{t-1})$
  $= \quad 1 \quad + \quad 0 \quad + \quad 0 \quad = 1$

- $Var(y_t) = Var(1 + \varepsilon_t + \varepsilon_{t-1})$
  $= Var(1) + Var(\varepsilon_t) + Var(\varepsilon_{t-1})$
  $= \quad 0 \quad + \quad \sigma_\varepsilon^2 \quad + \sigma_\varepsilon^2 = 2\sigma_\varepsilon^2$

- $\gamma_k = Cov(y_t, y_{t-k})$ $\qquad \boxed{Cov(X,Y) = E\left[\left(X-\mu_X\right)\left(Y-\mu_Y\right)\right] = E(XY) - \mu_X\mu_Y}$

$$\gamma_k = Cov(y_t, y_{t-k}) = E(y_t y_{t-k}) - E(y_t)E(y_{t-k})$$
$$= E[y_t y_{t-k}] - 1$$
$$= E[(1 + \varepsilon_t + \varepsilon_{t-1})(1 + \varepsilon_{t-k} + \varepsilon_{t-k-1})] - 1$$

$$= E[1 + \varepsilon_t + \varepsilon_{t-1} + \varepsilon_{t-k} + \varepsilon_t\varepsilon_{t-k} + \varepsilon_{t-1}\varepsilon_{t-k} + \varepsilon_{t-k-1} + \varepsilon_t\varepsilon_{t-k-1} + \varepsilon_{t-1}\varepsilon_{t-k-1}] - 1$$

$$\boxed{E(\varepsilon_{t-k}\varepsilon_{t-k}) = \sigma_\varepsilon^2 \qquad ; k = 0,1,2,3 \ldots}$$

$\boldsymbol{for\ k = 0}$:

$$\gamma_0 = E[1 + \varepsilon_t + \varepsilon_{t-1} + \varepsilon_t + \boxed{\varepsilon_t \varepsilon_t} + \varepsilon_{t-1}\varepsilon_t + \varepsilon_{t-1} + \varepsilon_t \varepsilon_{t-1} + \boxed{\varepsilon_{t-1}\varepsilon_{t-1}}] - 1$$

$$= 1 + \sigma_\varepsilon^2 + \sigma_\varepsilon^2 - 1 = 2\sigma_\varepsilon^2$$

$\boldsymbol{for\ k = 1}$:

$$\gamma_1 = E[1 + \varepsilon_t + \varepsilon_{t-1} + \varepsilon_{t-1} + \varepsilon_t \varepsilon_{t-1} + \boxed{\varepsilon_{t-1}\varepsilon_{t-1}} + \varepsilon_{t-2} + \varepsilon_t \varepsilon_{t-2} + \varepsilon_{t-1}\varepsilon_{t-2}] - 1$$

$$= 1 + \sigma_\varepsilon^2 - 1 = \sigma_\varepsilon^2$$

$\boldsymbol{for\ k \geq 2}$:

$$\gamma_k = E[1 + \varepsilon_t + \varepsilon_{t-1} + \varepsilon_{t-2} + \varepsilon_t \varepsilon_{t-2} + \varepsilon_{t-1}\varepsilon_{t-2} + \varepsilon_{t-3} + \varepsilon_t \varepsilon_{t-3} + \varepsilon_{t-1}\varepsilon_{t-3}] - 1$$

$$= 1 - 1 = 0$$

Thus, ACF for this process has the form:

$$\boxed{ACF = \rho_k = \frac{\gamma_k}{\gamma_0} = \frac{Cov(y_t, y_{t-k})}{Var(y_t)}}$$

$$\rho_k = \frac{\gamma_k}{\gamma_0} = \begin{cases} \frac{\gamma_0}{\gamma_0} = \frac{2\sigma_\varepsilon^2}{2\sigma_\varepsilon^2} = 1 & ;\ k = 0 \\ \frac{\gamma_1}{\gamma_0} = \frac{\sigma_\varepsilon^2}{2\sigma_\varepsilon^2} = 0.5 & ;\ k = 1 \\ \frac{\gamma_k}{\gamma_0} = \frac{0}{2\sigma_\varepsilon^2} = 0 & ;\ k \geq 2 \end{cases}$$

2- *Find the* ***ACF*** *for the process,* $Y_t = 1 + \varepsilon_t - \varepsilon_{t-1}$ *, plot it and compare it with the ACF in part (1).*

Solution:

$$ACF = \rho_k = \frac{\gamma_k}{\gamma_0} = \frac{Cov(y_t, y_{t-k})}{Var(y_t)}$$

- $E(y_t) = 1$

- $Var(y_t) = Var(1 + \varepsilon_t - \varepsilon_{t-1})$

$$= Var(1) + Var(\varepsilon_t) + Var(\varepsilon_{t-1})$$

$$= 0 + \sigma_\varepsilon^2 + \sigma_\varepsilon^2 = 2\sigma_\varepsilon^2$$

- $\gamma_k = Cov(y_t, y_{t-k})$

$$Cov(X, Y) = E[(X - \mu_X)(Y - \mu_Y)] = E(XY) - \mu_X \mu_Y$$

$$\begin{aligned}\gamma_k = Cov(y_t, y_{t-k}) &= E(y_t y_{t-k}) - E(y_t)E(y_{t-k}) \\ &= E[y_t y_{t-k}] - 1 \\ &= E[(1 + \varepsilon_t - \varepsilon_{t-1})(1 + \varepsilon_{t-k} - \varepsilon_{t-k-1})] - 1\end{aligned}$$

$$= E[1 + \varepsilon_t - \varepsilon_{t-1} + \varepsilon_{t-k} + \varepsilon_t\varepsilon_{t-k} - \varepsilon_{t-1}\varepsilon_{t-k} - \varepsilon_{t-k-1} - \varepsilon_t\varepsilon_{t-k-1} + \varepsilon_{t-1}\varepsilon_{t-k-1}] - 1$$

$$E(\varepsilon_{t-k}\varepsilon_{t-k}) = \sigma_\varepsilon^2 \qquad ; k = 0,1,2,3 \ldots$$

$\boldsymbol{for\ k = 0}$:

$$\gamma_0 = E\left[1 + \varepsilon_t - \varepsilon_{t-1} + \varepsilon_t + \boxed{\varepsilon_t \varepsilon_t} - \varepsilon_{t-1}\varepsilon_t - \varepsilon_{t-1} - \varepsilon_t \varepsilon_{t-1} + \boxed{\varepsilon_{t-1}\varepsilon_{t-1}}\right] - 1$$

$$= 1 + \sigma_\varepsilon^2 + \sigma_\varepsilon^2 - 1 = 2\sigma_\varepsilon^2$$

$\boldsymbol{for\ k = 1}$:

$$\gamma_1 = E\left[1 + \varepsilon_t - \varepsilon_{t-1} + \varepsilon_{t-1} + \varepsilon_t \varepsilon_{t-1} - \boxed{\varepsilon_{t-1}\varepsilon_{t-1}} - \varepsilon_{t-2} - \varepsilon_t \varepsilon_{t-2} + \varepsilon_{t-1}\varepsilon_{t-2}\right] - 1$$

$$= 1 - \sigma_\varepsilon^2 - 1 = -\sigma_\varepsilon^2$$

$\boldsymbol{for\ k \geq 2}$:

$$\gamma_k = E[1 + \varepsilon_t - \varepsilon_{t-1} + \varepsilon_{t-2} + \varepsilon_t \varepsilon_{t-2} - \varepsilon_{t-1}\varepsilon_{t-2} - \varepsilon_{t-3} - \varepsilon_t \varepsilon_{t-3} + \varepsilon_{t-1}\varepsilon_{t-3}] - 1$$

$$= 1 - 1 = 0$$

Thus, the autocorrelation function for this process has the form:

$$\boxed{ACF = \rho_k = \frac{\gamma_k}{\gamma_0} = \frac{Cov(y_t, y_{t-k})}{Var(y_t)}}$$

$$\rho_k = \frac{\gamma_k}{\gamma_0} = \begin{cases} \frac{\gamma_0}{\gamma_0} = \frac{2\sigma_\varepsilon^2}{2\sigma_\varepsilon^2} = 1 & ;\ k = 0 \\ \frac{\gamma_1}{\gamma_0} = \frac{-\sigma_\varepsilon^2}{2\sigma_\varepsilon^2} = -0.5 & ;\ k = 1 \\ \frac{\gamma_k}{\gamma_0} = \frac{0}{2\sigma_\varepsilon^2} = 0 & ;\ k \geq 2 \end{cases}$$

We notice from the autocorrelation functions in (1) and (2) that in process in (1) observations that are one time lag apart are positively correlated with $\rho_1 = 0.5$*, and that observations more than one time lag apart are not correlated. While the process in part (2) observations that are one time lag apart are negatively correlated with* $\rho_1 = -0.5$*, and that observations more than one time lag apart are not correlated.*

Question 3:

If the series $\{Y_t\}$ *ca n be expressed in the form:*

$$Y_t = \beta_0 + \beta_1 t + \beta_2 t^2 + \varepsilon_t$$

where $\{\varepsilon_t\}$ *as defined as in Q.2.*

1- *Find the expectation, the variance and the **ACF** of the series.*

$$\boxed{ACF = \rho_k = \frac{\gamma_k}{\gamma_0} = \frac{Cov(y_t, y_{t-k})}{Var(y_t)}}$$

- $E(y_t) \quad = E(\beta_0 + \beta_1 t + \beta_2 t^2 + \varepsilon_t)$

  $= E(\beta_0) + E(\beta_1 t) + E(\beta_2 t^2) + E(\varepsilon_t)$

  $= \beta_0 + \beta_1 t + \beta_2 t^2 + 0$

- $Var(y_t) = Var(\beta_0 + \beta_1 t + \beta_2 t^2 + \varepsilon_t)$

  $= Var(\beta_0 + \beta_1 t + \beta_2 t^2) + Var(\varepsilon_t)$

  $= \quad 0 \quad + \quad \sigma_\varepsilon^2 \quad = \sigma_\varepsilon^2$

- *Because $f(t) = \beta_0 + \beta_1 t + \beta_2 t^2$ is a deterministic function and not random*

$$\gamma_{s,t} = Cov[(\beta_0 + \beta_1 t + \beta_2 t^2 + \varepsilon_t), (\beta_0 + \beta_1 s + \beta_2 s^2 + \varepsilon_s)]$$

$$\boxed{Cov(X, Y) = E[(X - \mu_X)(Y - \mu_Y)] = E(XY) - \mu_X \mu_Y}$$

$$= E[(\beta_0 + \beta_1 t + \beta_2 t^2 + \varepsilon_t - E(y_t))(\beta_0 + \beta_1 s + \beta_2 s^2 + \varepsilon_s - E(y_s))]$$

$$= E[(\beta_0 + \beta_1 t + \beta_2 t^2 + \varepsilon_t - \beta_0 - \beta_1 t - \beta_2 t^2)(\beta_0 + \beta_1 s + \beta_2 s^2 + \varepsilon_s - \beta_0 - \beta_1 s - \beta_2 s^2)]$$

$$= E[(\varepsilon_t)(\varepsilon_s)] = 0, \quad s \neq t$$

*Thus, the **ACF** for this process has the form:*

$$\boxed{ACF = \rho_k = \frac{\gamma_k}{\gamma_0} = \frac{Cov(y_t, y_{t-k})}{Var(y_t)}}$$

$$\rho_k = \frac{\gamma_k}{\gamma_0} = \begin{cases} \dfrac{\gamma_0}{\gamma_0} = \dfrac{\sigma_\varepsilon^2}{\sigma_\varepsilon^2} = 1 & ; k = 0 \\ \dfrac{\gamma_k}{\gamma_0} = \dfrac{0}{\sigma_\varepsilon^2} = 0 & ; k > 0 \end{cases}$$

2- *Does this series fulfill the stationarity conditions? Discuss.*

Since the mean function is a function of time t, then the process is <u>not stationary</u> although the variance function is not a function of time and that the ACF depends only on time lag.

Question 4:

The following data represent the total profit (in million riyals) for a company:

Year	1430	1431	1432	1433	1434	1435	1436	1437
Profit y_t	3	2	2	4	5	6.1	4.4	5.5

1- *Calculate the coefficients of the sample autocorrelation function (SACF)* r_k

solution:

$$\bar{y} = \frac{3+2+2+4+5+6.1+4.4+5.5}{8} = \frac{32}{8} = 4$$

$$r_k = \hat{\rho}(k) = \frac{\gamma_k}{\gamma_0} = \frac{\sum_{t=1}^{n-k}(y_t-\bar{y})(y_{t+k}-\bar{y})}{\sum_{t=1}^{n}(y_t-\bar{y})^2}$$

1- *We can easily find:*

Year	y_t	$(y_t - \bar{y})$	$(y_t - \bar{y})^2$
1430	3	-1	1
1431	2	-2	4
1432	2	-2	4
1433	4	0	0
1434	5	1	1
1435	6.1	2.1	4.41
1436	4.4	0.4	0.16
1437	5.5	1.5	2.25
	32		16.82

$$\sum_{t=1}^{8}(y_t - 4)^2 = 16.82$$

Year	$(y_t - \bar{y})$	$(y_t - \bar{y})(y_{t+1} - \bar{y})$	$(y_t - \bar{y})(y_{t+2} - \bar{y})$	$(y_t - \bar{y})(y_{t+3} - \bar{y})$	$(y_t - \bar{y})(y_{t+4} - \bar{y})$	$(y_t - \bar{y})(y_{t+5} - \bar{y})$	$(y_t - \bar{y})(y_{t+6} - \bar{y})$	$(y_t - \bar{y})(y_{t+7} - \bar{y})$
1430	-1	2	2	0	-1	-2.1	-0.4	-1.5
1431	-2	4	0	-2	-4.2	-0.8	-3	
1432	-2	0	-2	-4.2	-0.8	-3		
1433	0	0	0	0	0			
1434	1	2.1	0.4	1.5				
1435	2.1	0.84	3.15					
1436	0.4	0.6						
1437	1.5							
$\sum_{t=1}^{n-k}(y_t - \bar{y})(y_{t+k} - \bar{y})$		9.54	3.55	-4.7	-6	-5.9	-3.4	-1.5
$\sum_{t=1}^{8}(y_t - 4)^2$		16.82	16.82	16.82	16.82	16.82	16.82	16.82
$r_k = \hat{\rho}(k)$		0.567	0.211	-0.279	-0.369	-0.351	-0.202	-0.089

According to the definition of SACF r_k then:

$$r_1 = \hat{\rho}(1) = \frac{\sum_{t=1}^{7}(y_t - 4)(y_{t+1} - 4)}{16.82} = 0.5672$$

$$r_2 = \hat{\rho}(2) = \frac{\sum_{t=1}^{6}(y_t - 4)(y_{t+2} - 4)}{16.82} = 0.2111$$

$$r_3 = \hat{\rho}(3) = \frac{\sum_{t=1}^{5}(y_t - 4)(y_{t+3} - 4)}{16.82} = -0.2794$$

$$r_4 = \hat{\rho}(4) = \frac{\sum_{t=1}^{4}(y_t - 4)(y_{t+4} - 4)}{16.82} = -0.3567$$

$$r_5 = \hat{\rho}(5) = \frac{\sum_{t=1}^{3}(y_t - 4)(y_{t+5} - 4)}{16.82} = -0.3508$$

$$r_6 = \hat{\rho}(6) = \frac{\sum_{t=1}^{2}(y_t - 4)(y_{t+6} - 4)}{16.82} = -0.2021$$

$$r_7 = \hat{\rho}(7) = \frac{\sum_{t=1}^{1}(y_t - 4)(y_{t+7} - 4)}{16.82} = -0.0892$$

also, we can plot the SACF by two axes: x-axis having the lag times between the observations, and the y-axes: the corresponding SACF coefficients, the resulting figure is called the correlogram.

2- *Calculate the standard errors for these estimates.*

We can calculate the SE of r_k from Bartlet's equation:

$$SE(r_k) \cong \sqrt{\frac{1}{n}\left(1 + 2\sum_{j=1}^{q} r_k^2\right)} \quad , k > q$$

$$SE(r_1) \cong \sqrt{\frac{1}{8}(1)} = 0.3536$$

$$SE(r_2) \cong \sqrt{\frac{1}{8}(1 + 2\, r_1^2)} = 0.4532$$

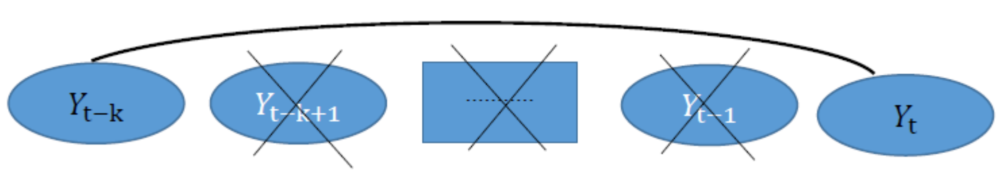
$$SE(r_3) \cong \sqrt{\frac{1}{8}(1 + 2\, r_1^2 + 2\, r_2^2)} = 0.4654$$

$$SE(r_4) \cong \sqrt{\frac{1}{8}(1 + 2\, r_1^2 + 2\, r_2^2 + 2\, r_3^2)} = \cdots$$

$$SE(r_5) \cong \sqrt{\frac{1}{8}(1 + 2\, r_1^2 + 2\, r_2^2 + 2\, r_3^2 + 2\, r_4^2)} = \cdots$$

$$SE(r_6) \cong \sqrt{\frac{1}{8}(1 + 2\, r_1^2 + 2\, r_2^2 + 2\, r_3^2 + 2\, r_4^2 + 2\, r_5^2)} = \cdots$$

$$SE(r_7) \cong \sqrt{\frac{1}{8}(1 + 2\, r_1^2 + 2\, r_2^2 + 2\, r_3^2 + 2\, r_4^2 + 2\, r_5^2 + 2\, r_6^2)} = \cdots$$



3- *Find Z value of $\boldsymbol{\rho_k}$ for k=1,2 and test the significant using α =0.05*

$Z = \frac{r_k}{SE(r_k)}$

$$H_0: \rho_k = 0 \qquad H_1 = \rho_k \neq 0$$

We reject the null hypothesis, at significance level α *if* $|z| > z_{\alpha/2}$

$$Z_1 = \frac{0.5672}{0.3536} = 1.6041 < 1.96 \quad we\ fail\ to\ reject\ H_0\ , so\ \rho_1 = 0$$

$$Z_2 = \frac{0.2111}{0.4532} = 0.4568 < 1.96 \quad we\ fail\ to\ reject\ H_0\ , so\ \rho_2 = 0$$

Question 5:

Write the Yule-Walker equations for every model of the following, where $\varepsilon_t \sim i.i.d \sim N(0, \sigma_\varepsilon^2)$ *:*

1- $\boldsymbol{y_t = 0.5 y_{t-1} + \varepsilon_t}$ ***(model 1)***

- *to find Yule-Walker equations for the model:*

first, multiply both sides by y_{t-k}.

second, taking the expectation.

finally, dividing both sides by γ_0

first, $\quad y_t y_{t-k} = 0.5 y_{t-1} y_{t-k} + \varepsilon_t y_{t-k}$

second, $\quad E(y_t y_{t-k}) = 0.5\, E(y_{t-1} y_{t-k}) + E(\varepsilon_t y_{t-k})$

$$\gamma_k = 0.5\, \gamma_{k-1} + 0 \qquad for\ k \neq 0$$

finally, $\quad \frac{\gamma_k}{\gamma_0} = 0.5\, \frac{\gamma_{k-1}}{\gamma_0} \qquad \boxed{\rho_k = \frac{\gamma_k}{\gamma_0}}$

$$\rho_k = 0.5\, \rho_{k-1} \qquad , \qquad k = 1,2,3, \ldots$$

These equations are called Yule-Walker equations, we can use them in finding autocorrelation and partial autocorrelation coefficients of the model.

2- $\boldsymbol{Y_t = 1.2\, y_{t-1} - 0.7\, y_{t-2} + \varepsilon_t}$ ***(model 2)***

- *to find Yule-Walker equations for the model:*

first, multiply both sides by y_{t-k}.

second, taking the expectation.

finally, dividing both sides by γ_0

first, $y_t y_{t-k} = 1.2\, y_{t-1} y_{t-k} - 0.7\, y_{t-2} y_{t-k} + \varepsilon_t y_{t-k}$

second, $E(y_t y_{t-k}) = 1.2\, E(y_{t-1} y_{t-k}) - 0.7\, E(y_{t-2} y_{t-k}) + E(\varepsilon_t y_{t-k})$

$$\gamma_k = 1.2\, \gamma_{k-1} - 0.7\, \gamma_{k-2} \qquad for\ k \neq 0$$

finally, $\frac{\gamma_k}{\gamma_0} = 1.2\, \frac{\gamma_{k-1}}{\gamma_0} - 0.7\, \frac{\gamma_{k-2}}{\gamma_0}$ $\boxed{\rho_k = \frac{\gamma_k}{\gamma_0}}$

$$\rho_k = 1.2\, \rho_{k-1} - 0.7 \rho_{k-2} \qquad , \qquad k = 1,2,3, \ldots$$

3- Find ρ_1, ρ_2 , ϕ_{kk} for the models in (1) and (2).

*For **model (1)** we found the following ACF:*

$$\rho_k = 0.5\,\rho_{k-1} \quad , \quad k = 1,2,3,\ldots$$

$$\boxed{\rho_0 = 1}$$

when k=1 $\quad \rho_1 = 0.5\,\rho_0 = 0.5(1) = 0.5$

when k=2 $\quad \rho_2 = 0.5\,\rho_1 = 0.5(0.5) = 0.25$

when k=3 $\quad \rho_3 = 0.5\,\rho_2 = 0.5(0.25) = 0.125$

Applying the recurrence relation for finding the PACF:

$$\varphi_{00} = 1 \quad , \quad \varphi_{11} = \rho_1$$

$$\varphi_{kk} = \frac{\rho_k - \sum_{j=1}^{k-1} \varphi_{k-1,j}\rho_{k-j}}{1 - \sum_{j=1}^{k-1} \varphi_{k-1,j}\rho_j}$$

$$\varphi_{kj} = \varphi_{k-1,j} - \varphi_{kk}\varphi_{k-1,k-j} \quad ,j=1,2,\ldots,k-1$$

$$\phi_{00} = 1$$

$$\phi_{11} = \rho_1 = 0.5$$

$$\phi_{22} = \frac{\rho_2 - \sum_{j=1}^{1} \phi_{1,j}\ \rho_{2-j}}{1-\sum_{j=1}^{1} \phi_{1,j}\ \rho_j} = \frac{\rho_2 - \phi_{11}\rho_1}{1-\phi_{11}\rho_1} = \frac{0.25-(0.5)(0.5)}{1-(0.5)(0.5)} = 0$$

$$\boxed{\phi_{21} = \phi_{11} - \phi_{22}\phi_{11} = 0.5 - (0)(0.5) = 0.5}$$

$$\phi_{33} = \frac{\rho_3 - \sum_{j=1}^{2} \phi_{2,j}\ \rho_{3-j}}{1 - \sum_{j=1}^{2} \phi_{2,j}\ \rho_j} = \frac{\rho_3 - \phi_{21}\rho_2 - \phi_{22}\rho_1}{1 - \phi_{21}\rho_1 - \phi_{22}\rho_2} = \frac{0.125 - 0.5 \times 0.25 - 0 \times 0.5}{1 - \phi_{21}\rho_1 - \phi_{22}\rho_2} = 0$$

$$similarly,\ \ \phi_{44} = \phi_{55} = \cdots = 0$$

*For **model (2)** we found the following ACF:*

$$\rho_k = 1.2\ \rho_{k-1} - 0.7\rho_{k-2},\ \ k = 1,2,3,\ldots$$

$$\boxed{\rho_0 = 1}$$

when k=1 $\quad \rho_1 = 1.2\ \rho_0 - 0.7\rho_1 \qquad \boxed{\rho_{-1} = \rho_1}$

$$\rho_1 = 1.2 - 0.7\rho_1$$

$$\rho_1\ (1 + 0.7) = 1.2 \ \Longrightarrow \rho_1 = \frac{1.2}{1.7} \quad \Longrightarrow \boxed{\rho_1 = 0.7059}$$

when k=2 $\quad \rho_2 = 1.2\ \rho_1 - 0.7\rho_0$

$$\rho_2 = 1.2\ (0.7059) - 0.7(1) \qquad \Longrightarrow \boxed{\rho_2 = 0.1471}$$

when k=3 $\quad \rho_3 = 1.2\ \rho_2 - 0.7\rho_1$

$$\rho_2 = 1.2\ (0.1471) - 0.7(0.7059) \Longrightarrow \boxed{\rho_3 = -0.3176}$$

Applying the recurrence relation for finding the PACF:

$$\varphi_{00} = 1 \quad , \quad \varphi_{11} = \rho_1$$

$$\varphi_{kk} = \frac{\rho_k - \sum_{j=1}^{k-1} \varphi_{k-1,j}\rho_{k-j}}{1 - \sum_{j=1}^{k-1} \varphi_{k-1,j}\rho_j}$$

$$\varphi_{kj} = \varphi_{k-1,j} - \varphi_{kk}\varphi_{k-1,k-j} \quad ,j=1,2,\dots,k-1$$

$$\phi_{00} = 1$$

$$\phi_{11} = \rho_1 = 0.7059$$

$$\phi_{22} = \frac{\rho_2 - \sum_{j=1}^{1} \phi_{1,j}\ \rho_{2-j}}{1-\sum_{j=1}^{1} \phi_{1,j}\ \rho_j} = \frac{\rho_2 - \phi_{11}\rho_1}{1-\phi_{11}\rho_1} = \frac{0.1471-(0.7059)(0.7059)}{1-(0.7059)(0.7059)} = -0.7$$

$$\phi_{33} = \frac{\rho_3 - \sum_{j=1}^{2} \phi_{2,j}\ \rho_{3-j}}{1-\sum_{j=1}^{2} \phi_{2,j}\ \rho_j} = \frac{\rho_3 - [\phi_{21}\rho_2 + \phi_{22}\rho_1]}{1-[\phi_{21}\rho_1 + \phi_{22}\rho_2]}$$

$$\boxed{\phi_{21} = \phi_{11} - \phi_{22}\phi_{11} = 0.7059 - (-0.7)(0.7059) = 1.2}$$

$$\phi_{33} = \frac{-0.1376-[1.2\,(0.1471)+(-0.7)(0.7059)]}{1-[1.2(0.7059)+(-0.7)(0.1471)]} = 0$$

similarly, $\phi_{44} = \phi_{55} = \cdots = 0$

Question 6:

Find the Yule-Walker equations for the following models and solve these equations to get values for ρ_1 and ρ_2.

1- $\boldsymbol{y_t - 0.8\, y_{t-1} = \varepsilon_t}$

first, $y_t y_{t-k} - 0.8\, y_{t-1} y_{t-k} = \varepsilon_t\, y_{t-k}$

second, $E(y_t y_{t-k}) = 0.8\, E(y_{t-1} y_{t-k}) + E(\varepsilon_t y_{t-k})$

$$\gamma_k = 0.8\, \gamma_{k-1} + 0 \qquad for\ k \neq 0$$

finally, $\frac{\gamma_k}{\gamma_0} = 0.8\, \frac{\gamma_{k-1}}{\gamma_0}$ $\qquad \boxed{\rho_k = \frac{\gamma_k}{\gamma_0}}$

$$\rho_k = 0.8\, \rho_{k-1} \quad , \quad k = 1,2,3,\ldots$$

when k=1 $\rho_1 = 0.8\, \rho_0$

$$\rho_1 = 0.8(1) \quad \Longrightarrow \boxed{\rho_1 = 0.8}$$

when k=2 $\rho_2 = 0.8\, \rho_1$

$$\rho_2 = 0.8\,(0.8) \quad \Longrightarrow \boxed{\rho_2 = 0.64}$$

2- $\boldsymbol{y_t = 0.9\, y_{t-1} - 0.4\, y_{t-2} + \varepsilon_t}$

first, $y_t y_{t-k} = 0.9\, y_{t-1} y_{t-k} - 0.4\, y_{t-2} y_{t-k} + \varepsilon_t y_{t-k}$

second, $E(y_t y_{t-k}) = 0.9\, E(y_{t-1} y_{t-k}) - 0.4\, E(y_{t-2} y_{t-k}) + E(\varepsilon_t y_{t-k})$

$$\gamma_k = 0.9\, \gamma_{k-1} - 0.4\, \gamma_{k-2} + 0 \qquad for\ k \neq 0$$

finally, $\frac{\gamma_k}{\gamma_0} = 0.9\, \frac{\gamma_{k-1}}{\gamma_0} - 0.4\, \frac{\gamma_{k-2}}{\gamma_0}$ $\boxed{\rho_k = \frac{\gamma_k}{\gamma_0}}$

$$\rho_k = 0.9\, \rho_{k-1} - 0.4\, \rho_{k-2} \quad , \quad k = 1,2,3, \dots$$

when k=1 $\rho_1 = 0.9\, \rho_0 - 0.4\, \rho_1$ $\boxed{\rho_{-1} = \rho_1}$

$$\rho_1 = 0.9(1) - 0.4\, \rho_1$$

$$\rho_1 + 0.4\, \rho_1 = 0.9$$

$$\rho_1 (1 + 0.4\,) = 0.9$$

$$\rho_1 = \frac{0.9}{(1+0.4\,)} \quad \Longrightarrow \boxed{\rho_1 = 0.643}$$

when k=2 $\rho_2 = 0.9\, \rho_1 - 0.4\, \rho_0$

$$\rho_2 = 0.9(0.643) - 0.4(1) \Longrightarrow \boxed{\rho_2 = 0.1787}$$

Question 7:

Assume $\varepsilon_t \sim i.i.d \sim N(0, \sigma_\varepsilon^2)$ *, and let the observed series be defined as*

$$y_t = \varepsilon_t - \theta \varepsilon_{t-1}$$

Where the parameter θ *can take either the value* $\theta = 3$ *or* $\theta = \frac{1}{3}$.

1- *Find the autocorrelation function of the series* $\{Y_t\}$ *for both cases, compare them.*

$$\boxed{for\ \theta = 3 \quad \longrightarrow \quad y_t = \varepsilon_t - 3\varepsilon_{t-1}}$$

- $E(y_t) = E(\varepsilon_t - 3\varepsilon_{t-1})$

  $= E(\varepsilon_t) + E(-3\varepsilon_{t-1}) = 0$

- $Var(y_t) = Var(\varepsilon_t - 3\varepsilon_{t-1})$

  $= Var(\varepsilon_t) + 9Var(\varepsilon_{t-1})$

  $= \sigma_\varepsilon^2 + 9\sigma_\varepsilon^2 = 10\sigma_\varepsilon^2$

- $\gamma_k = Cov[y_t, y_{t-k}]$

  $= Cov[(\varepsilon_t - 3\varepsilon_{t-1}), (\varepsilon_{t-k} - 3\varepsilon_{t-k-1})]$

for k=1:

$$\gamma_1 = Cov[(\varepsilon_t - 3\varepsilon_{t-1}), (\varepsilon_{t-1} - 3\varepsilon_{t-2})]$$

$$\boxed{Cov(X,Y) = E\left[\left(X - \mu_X\right)\left(Y - \mu_Y\right)\right] = E(XY) - \mu_X\mu_Y}$$

$$= E[(\varepsilon_t - 3\varepsilon_{t-1})(\varepsilon_{t-1} - 3\varepsilon_{t-2})]$$

$$= E\left[\varepsilon_t\varepsilon_{t-1}\boxed{-3\varepsilon_{t-1}\varepsilon_{t-1}} + 3\varepsilon_t\varepsilon_{t-2} - 9\varepsilon_{t-1}\varepsilon_{t-2}\right] = -3\ \sigma_\varepsilon^2$$

for k=2:

$$\gamma_2 = Cov[(\varepsilon_t - 3\varepsilon_{t-1}), (\varepsilon_{t-2} - 3\varepsilon_{t-3})] = 0$$

$$\boxed{Cov(X,Y) = E\left[\left(X - \mu_X\right)\left(Y - \mu_Y\right)\right] = E(XY) - \mu_X\mu_Y}$$

$$= E[(\varepsilon_t - 3\varepsilon_{t-1})(\varepsilon_{t-2} - 3\varepsilon_{t-3})] = 0$$

Hence, the ACF has the form:

$$\rho_k = \frac{\gamma_k}{\gamma_0} = \begin{cases} \frac{\gamma_0}{\gamma_0} = \frac{10\ \sigma_\varepsilon^2}{10\ \sigma_\varepsilon^2} = 1 & ,\ k = 0 \\ \frac{\gamma_1}{\gamma_0} = \frac{-3\ \sigma_\varepsilon^2}{10\ \sigma_\varepsilon^2} = -0.3 & ,\ k = 1 \\ 0 & ,\ k \geq 2 \end{cases}$$

Thus, we notice that both process has the same ACF!

$$\boxed{for\ \theta = \frac{1}{3} \quad \longrightarrow \quad y_t = \varepsilon_t - \frac{1}{3}\varepsilon_{t-1}}$$

- $E(y_t) = E\left(\varepsilon_t - \frac{1}{3}\varepsilon_{t-1}\right)$

$$= E(\varepsilon_t) + E\left(-\frac{1}{3}\varepsilon_{t-1}\right) = 0$$

- $Var(y_t) = Var\left(\varepsilon_t - \frac{1}{3}\varepsilon_{t-1}\right)$

$$= Var(\varepsilon_t) + \frac{1}{9}Var(\varepsilon_{t-1})$$

$$= \quad \sigma_\varepsilon^2 \quad + \quad \frac{1}{9} \quad \sigma_\varepsilon^2 \quad = \frac{10}{9}\ \sigma_\varepsilon^2$$

- $\gamma_k = Cov[y_t, y_{t-k}]$

$$= Cov\left[\left(\varepsilon_t - \frac{1}{3}\varepsilon_{t-1}\right), \left(\varepsilon_{t-k} - \frac{1}{3}\varepsilon_{t-k-1}\right)\right]$$

for k=1:

$$\gamma_1 = Cov\left[\left(\varepsilon_t - \frac{1}{3}\varepsilon_{t-1}\right), \left(\varepsilon_{t-1} - \frac{1}{3}\varepsilon_{t-2}\right)\right]$$

$$\boxed{Cov(X,Y) = E\left[\left(X-\mu_X\right)\left(Y-\mu_Y\right)\right] = E(XY) - \mu_X\mu_Y}$$

$$= E\left[\left(\varepsilon_t - \frac{1}{3}\varepsilon_{t-1}\right)\left(\varepsilon_{t-1} - \frac{1}{3}\varepsilon_{t-2}\right)\right]$$

$$= E\left[\varepsilon_t\varepsilon_{t-1} \boxed{-\frac{1}{3}\varepsilon_{t-1}\varepsilon_{t-1}} + \frac{1}{3}\varepsilon_t\varepsilon_{t-2} - \frac{1}{9}\varepsilon_{t-1}\varepsilon_{t-2}\right] = -\frac{1}{3}\ \sigma_\varepsilon^2$$

for k=2:

$$\gamma_2 = Cov\left[\left(\varepsilon_t - \frac{1}{3}\varepsilon_{t-1}\right), \left(\varepsilon_{t-2} - \frac{1}{3}\varepsilon_{t-3}\right)\right] = 0$$

$$\boxed{Cov(X,Y) = E\left[\left(X-\mu_X\right)\left(Y-\mu_Y\right)\right] = E(XY) - \mu_X\mu_Y}$$

$$= E\left[\left(\varepsilon_t - \frac{1}{3}\varepsilon_{t-1}\right)\left(\varepsilon_{t-2} - \frac{1}{3}\varepsilon_{t-3}\right)\right] = 0$$

Hence, the ACF has the form:

$$\rho_k = \frac{\gamma_k}{\gamma_0} = \begin{cases} \dfrac{\gamma_0}{\gamma_0} = \dfrac{\frac{10}{9}\sigma_\varepsilon^2}{\frac{10}{9}\sigma_\varepsilon^2} = 1 & , \ k = 0 \\ \dfrac{\gamma_1}{\gamma_0} = \dfrac{-\frac{1}{3}\sigma_\varepsilon^2}{\frac{10}{9}\sigma_\varepsilon^2} = -0.3 & , \ k = 1 \\ 0 & , \ k \geq 2 \end{cases}$$

Thus we notice that both process has the same ACF!

2- *Is the process* $\{Y_t\}$ *stationary in both cases?*

yes

3- *For simplification, assume that the mean of the process* $\{Y_t\}$ *equal zero, and the variance is equal to one, and that you obtained the observed series* $\{Y_t\}$ *for* $t = 1,2,\dots,n$, *and that you have obtained a credible estimates for the coefficients of the ACF* ρ_k, *can you tell which process generated the data (i.e. which value* $\theta = 3$ *or* $\theta = \frac{1}{3}$ *to be used in the model to model the data?)*

Question 8:

Write the following models using the backshift operator B:

3- $y_t - 0.5\, y_{t-1} = \varepsilon_t$:

By_t → $y_{t-k} = B^k y_t$

$$y_t - 0.5\, By_t = \varepsilon_t$$

$$(1 - 0.5B)\, y_t = \varepsilon_t$$

4- $y_t = \varepsilon_t - 1.3\varepsilon_{t-1} + 0.4\, \varepsilon_{t-2}$

$\varepsilon_{t-1} = B\, \varepsilon_t$

$\varepsilon_{t-2} = B^2 \varepsilon_t$

$$y_t = \varepsilon_t - 1.3B\varepsilon_t + 0.4\, B^2 \varepsilon_t$$

$$y_t = (1 - 1.3\, B + 0.4B^2)\, \varepsilon_t$$

5- $y_t - 0.5\, y_{t-1} = \varepsilon_t - 1.3\, \varepsilon_{t-1} + 0.4\, \varepsilon_{t-2}$

$$\boxed{\begin{aligned} y_{t-1} &= By_t \\ \varepsilon_{t-1} &= B\,\varepsilon_t \\ \varepsilon_{t-2} &= B^2\varepsilon_t \end{aligned}}$$

$$y_t - 0.5\,By_t = \varepsilon_t - 1.3B\varepsilon_t + 0.4\,B^2\varepsilon_t$$

$$(1-0.5B)\,y_t = (1-1.3\,B+0.4B^2)\,\varepsilon_t$$

Question 9:

Express the following models in terms of the process $\{y_t\}$ *and* $\{\varepsilon_t\}$*:*

3- $\nabla^3 y_t = \nabla\varepsilon_t$

$\nabla\varepsilon_t = \varepsilon_t - \varepsilon_{t-1}$

$\nabla^3 y_t = \nabla\nabla\nabla(y_t)$

$= \nabla\nabla(y_t - y_{t-1})$

$= \nabla[(y_t - y_{t-1}) - (y_{t-1} - y_{t-2})]$

$= \nabla[y_t - y_{t-1} - y_{t-1} + y_{t-2}]$

$= \nabla(y_t - 2y_{t-1} + y_{t-2})$

$= (y_t - 2y_{t-1} + y_{t-2}) - (y_{t-1} - 2y_{t-2} + y_{t-3})$

$= y_t - 2y_{t-1} + y_{t-2} - y_{t-1} + 2y_{t-2} - y_{t-3}$

$= y_t - 3y_{t-1} + 3y_{t-2} - y_{t-3}$

$$y_t - 3y_{t-1} + 3y_{t-2} - y_{t-3} = \varepsilon_t - \varepsilon_{t-1}$$

4- $\nabla^2 y_t = \nabla^3 \varepsilon_t$

$$\nabla^3 \varepsilon_t = \nabla\nabla\nabla(\varepsilon_t)$$

$$= \nabla\nabla(\varepsilon_t - \varepsilon_{t-1})$$

$$= \nabla[(\varepsilon_t - \varepsilon_{t-1}) - (\varepsilon_{t-1} - \varepsilon_{t-2})]$$

$$= \nabla[\varepsilon_t - \varepsilon_{t-1} - \varepsilon_{t-1} + \varepsilon_{t-2}]$$

$$= \nabla[\varepsilon_t - 2\varepsilon_{t-1} + \varepsilon_{t-2}]$$

$$= [\varepsilon_t - 2\varepsilon_{t-1} + \varepsilon_{t-2}] - [\varepsilon_{t-1} - 2\varepsilon_{t-2} + \varepsilon_{t-3}]$$

$$= \varepsilon_t - 2\varepsilon_{t-1} + \varepsilon_{t-2} - \varepsilon_{t-1} + 2\varepsilon_{t-2} - \varepsilon_{t-3}$$

$$= \varepsilon_t - 3\varepsilon_{t-1} + 3\varepsilon_{t-2} - \varepsilon_{t-3}$$

$$\nabla^2 y_t = \nabla\nabla(y_t)$$

$$= \nabla(y_t - y_{t-1})$$

$$= [(y_t - y_{t-1}) - (y_{t-1} - y_{t-2})]$$

$$= [y_t - y_{t-1} - y_{t-1} + y_{t-2}]$$

$$= y_t - 2y_{t-1} + y_{t-2}$$

$$y_t - 2y_{t-1} + y_{t-2} = \varepsilon_t - 3\varepsilon_{t-1} + 3\varepsilon_{t-2} - \varepsilon_{t-3}$$

Question 10:

if the series $\{y_t\}$ *can be expressed in the for:*

$$y_t = \beta_0 + \beta_1 t + \beta_2 t^2 + \varepsilon_t$$

1. *Show that* $\{y_t\}$ *is not stationary process*

$$\mu_t = E(y_t) = E(\beta_0 + \beta_1 t + \beta_2 t^2 + \varepsilon_t)$$

$$= \beta_0 + \beta_1 t + \beta_2 t^2$$

μ_t *depend on time then* $\{y_t\}$ *is not stationary process*

2. *Use the difference operator* ∇^r *to render* $\{y_t\}$ *to a stationary process.*

$$\nabla y_t = z_t = y_t - y_{t-1}$$

$$= (\beta_0 + \beta_1 t + \beta_2 t^2 + \varepsilon_t) - (\beta_0 + \beta_1 (t-1) + \beta_2 (t-1)^2 + \varepsilon_{t-1})$$

$$= \beta_0 + \beta_1 t + \beta_2 t^2 + \varepsilon_t - \beta_0 - \beta_1 (t-1) - \beta_2 (t-1)^2 - \varepsilon_{t-1}$$

$$= \beta_0 + \beta_1 t + \beta_2 t^2 + \varepsilon_t - \beta_0 - \beta_1 t + \beta_1 - \beta_2 (t^2 - 2t + 1) - \varepsilon_{t-1}$$

$$= \beta_0 + \beta_1 t + \beta_2 t^2 + \varepsilon_t - \beta_0 - \beta_1 t + \beta_1 - \beta_2 t^2 + 2\beta_2 t - \beta_2 - \varepsilon_{t-1}$$

$$= \qquad + \varepsilon_t \qquad \beta_1 \qquad + 2\beta_2 t - \beta_2 - \varepsilon_{t-1}$$

$$= \beta_1 - \beta_2 + 2\beta_2 t + \varepsilon_t - \varepsilon_{t-1}$$

- $E(z_t) = E(\beta_1 - \beta_2 + 2\beta_2 t + \varepsilon_t - \varepsilon_{t-1}) = \beta_1 - \beta_2 + 2\beta_2 t$

  $E(z_t)$ *depend on time then it is also not stationary process*

$\nabla\nabla y_t = \nabla z_t = w_t$

$= (\beta_1 - \beta_2 + 2\beta_2 t + \varepsilon_t - \varepsilon_{t-1}) - (\beta_1 - \beta_2 + 2\beta_2(t-1) + \varepsilon_{t-1} - \varepsilon_{t-2})$

$= \beta_1 - \beta_2 + 2\beta_2 t + \varepsilon_t - \varepsilon_{t-1} - \beta_1 + \beta_2 - 2\beta_2(t-1) - \varepsilon_{t-1} + \varepsilon_{t-2}$

$= \beta_1 - \beta_2 + 2\beta_2 t + \varepsilon_t - \varepsilon_{t-1} - \beta_1 + \beta_2 - 2\beta_2 t + 2\beta_2 - \varepsilon_{t-1} + \varepsilon_{t-2}$

$= \quad \varepsilon_t - \varepsilon_{t-1} \quad + 2\beta_2 - \varepsilon_{t-1} + \varepsilon_{t-2}$

$= 2\beta_2 + \varepsilon_t - 2\varepsilon_{t-1} + \varepsilon_{t-2}$

- $E(w_t) = E(2\beta_2 + \varepsilon_t - 2\varepsilon_{t-1} + \varepsilon_{t-2}) = 2\beta_2$

  $E(w_t)$ do not depend on time.

- $Var(w_t) = Var(2\beta_2 + \varepsilon_t - 2\varepsilon_{t-1} + \varepsilon_{t-2}) = \sigma^2 + 4\sigma^2 + \sigma^2 = 6\sigma^2$

  $Var(w_t)$ do not depend on time.

- $\gamma_1 = Cov(w_t, w_{t-1})$

$$Cov(X,Y) = E\left[(X - \mu_X)(Y - \mu_Y)\right] = E(XY) - \mu_X\mu_Y$$

$\gamma_1 = Cov(w_t, w_{t-1})$

$= E\left[\left(w_t - E(w_t)\right)\left(w_{t-1} - E(w_{t-1})\right)\right]$

$= E[(2\beta_2 + \varepsilon_t - 2\varepsilon_{t-1} + \varepsilon_{t-2} - 2\beta_2)(2\beta_2 + \varepsilon_{t-1} - 2\varepsilon_{t-2} + \varepsilon_{t-3} - 2\beta_2)]$

$= E[(\varepsilon_t - 2\varepsilon_{t-1} + \varepsilon_{t-2})(\varepsilon_{t-1} - 2\varepsilon_{t-2} + \varepsilon_{t-3})]$

$= -2\sigma^2 - 2\sigma^2 = -4\sigma^2$

γ_1 *do not depend on time.*

then, $\nabla\nabla y_t$ *is stationary process.*

3. *Drive the autocorrelation function* ρ_k *for the stationary process you found in part 2.*

$$\gamma_2 = Cov(w_t, w_{t-2})$$

$$= E\left[\left(w_t - E(w_t)\right)\left(w_{t-2} - E(w_{t-2})\right)\right]$$

$$= E[(2\beta_2 + \varepsilon_t - 2\varepsilon_{t-1} + \varepsilon_{t-2} - 2\beta_2)(2\beta_2 + \varepsilon_{t-2} - 2\varepsilon_{t-3} + \varepsilon_{t-4} - 2\beta_2)]$$

$$= E[(\varepsilon_t - 2\varepsilon_{t-1} + \varepsilon_{t-2})(\varepsilon_{t-2} - 2\varepsilon_{t-3} + \varepsilon_{t-4})]$$

$$= E[(\varepsilon_{t-2})(\varepsilon_{t-2})]$$

$$= \sigma^2$$

$$\gamma_3 = \gamma_4 = \cdots = 0$$

thus, $\rho_k = \frac{\gamma_k}{\gamma_0}$

$$\rho_k = \frac{\gamma_k}{\gamma_0} = \begin{cases} \frac{\gamma_0}{\gamma_0} = 1 & k = 0 \\ \frac{\gamma_k}{\gamma_0} = -\frac{4\sigma^2}{6\sigma^2} = -\frac{4}{6} & k = 1 \\ \frac{\gamma_k}{\gamma_0} = \frac{\sigma^2}{6\sigma^2} = \frac{1}{6} & k = 2 \\ 0 & k \geq 3 \end{cases}$$