Introduction:

$$Mean: \ \bar{X} = \frac{\sum_{i=1}^{n} x_i}{n}$$

$$Variance: \ S^2 = \frac{\sum_{i=1}^{n}(x_i - \bar{X})^2}{n-1} = S^2 = \frac{\sum_{i=1}^{n} x_i^2 - n\bar{X}^2}{n-1}$$

$$Standard\ deviation: \ S = \sqrt{S^2}$$

Example:

Compute the sample mean, variance and standard deviation of the following observations (ages in year): 10, 21, 33, 53, and 54.

1. $Mean: \ \bar{X} = \frac{\sum_{i=1}^{n} x_i}{n}$

$$= \frac{10 + 21 + 33 + 53 + 54}{5} = \frac{171}{5} = 34.2\ years$$

2. $Variance: \ S^2 = \frac{\sum_{i=1}^{n}(x_i - \bar{X})^2}{n-1}$

$$= \frac{(10 - 34.2)^2 + (21 - 34.2)^2 + (33 - 34.2)^2 + (53 - 34.2)^2 + (54 - 34.2)^2}{5 - 1}$$

$$= 376.7\ year^2$$

3. $Standard\ deviation: \ S = \sqrt{\frac{\sum_{i=1}^{n}(x_i - \bar{X})^2}{n-1}}$

$$= \sqrt{S^2} = \sqrt{376.7} = 19.41\ years$$

Combinations &Permutations:

$$nCr = \binom{n}{r} = \frac{n!}{r!\,(n-r)!} \quad , \quad nPr = \frac{n!}{(n-r)!}$$

Solution of Q1 - p.1:

(a): $\binom{6}{4} = \frac{6!}{4!(6-4)!} = 15$

(b): $\binom{6}{2} = \frac{6!}{2!(6-2)!} = 15$

Solution of Q2 - P.1:

$$\binom{n}{n-x} = \frac{n!}{(n-x)!\,\big(n-(n-x)\big)!} = \frac{n!}{(n-x)!\,(n-n+x)!} = \frac{n!}{(n-x)!\,x!}$$

$$= \binom{n}{x}$$

Solution of Q3 - p.1:

(a): $\binom{n}{0} = \frac{n!}{0!(n-0)!} = 1$

(b): $\binom{n}{1} = \frac{n!}{1!(n-1)!} = n$

(c): $\binom{n}{n} = \frac{n!}{n!(n-n)!} = 1$

Solution of Q4 - p.1:

$$\binom{6}{3} = \frac{6!}{3!\,(6-3)!} = 20$$

Solution of Q5 - p.1:

$$\binom{5}{2} = \frac{5!}{2!\,(5-2)!} = 10$$

Solution of Q6 - p.1:

$$_5P_2 = \frac{5!}{(5-2)!} = 20$$

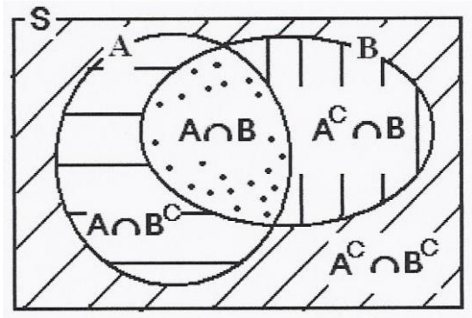
Probability:

Definitions and Theorems:

$*\ 0 \le P(A) \le 1$

$*\ P(S) = 1$

$*\ P(\emptyset) = 0$



1- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
2- $P(A|B) = P(A \cap B)/P(B)$
3- $P(A \cap B) = P(A) \times P(B)\ \ (if\ A\ \&\ B\ are\ independent.)$
4- $P(A \cap B) = 0\ \ (if\ A\ \&\ B\ are\ disjoint.)$
5- $P(A^c) = 1 - P(A) \qquad ; P(A^c) = P(\bar{A})$

Solution of Q1 - p.2:

Givens:

$$P(A) = 0.5, \quad P(B) = 0.4, \quad P(C \cap A^c) = 0.6,$$

$$P(C \cap A) = 0.2, \quad P(A \cup B) = 0.9$$

(a): $P(C) = P(C \cap A^c) + P(C \cap A) = 0.6 + 0.2 = 0.8$

(b): $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\Rightarrow \qquad 0.9 = 0.5 + 0.4 - P(A \cap B)$

$$P(A \cap B) = 0$$

(c):

$$P(C \mid A) = \frac{P(C \cap A)}{P(A)} = \frac{0.2}{0.5} = 0.4$$

(d): $P(B^c \cap A^c) = 1 - P(B \cup A) = 1 - 0.9 = 0.1$

Solution of Q2 - p.2:

(a): $n(S) = 2^3 = 8$, (Cause we have 2 outcomes and 3 trails)

(b): $S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$

$$Exactly\ 2\ heads = \{HHT, HTH, THH\}$$

$$P(Exactly\ 2\ heads) = \frac{3}{8}$$

(c):

$Exactly\ 2\ heads = \{HHT, HTH, THH\}$ & $Exactly\ 3\ heads = \{HHH\}$, We can see there is no relation between these two events, and hence, they are disjoint.

(d):

$$A = The\ first\ coin\ is\ head = \{HHH, HHT, HTH, HTT\}$$

$$B = Second\ and\ third\ tails = \{HTT, TTT\}$$

$$P(A) = \frac{4}{8} = 0.5 \quad \& \quad P(B) = \frac{2}{8} = 0.25$$

$$A \cap B = \{HTT\} \quad \& \quad P(A \cap B) = \frac{1}{8} = 0.125$$

We can see that,

$$P(A \cap B) = P(A) \times P(B)$$

$$0.125 = 0.5 \times 0.25$$

Therefore, A and B are independent.

Solution of Q3 - p.2:

The table below, summarizes the Sample Space for throwing the dice twice:

D1 / D2	1	2	3	4	5	6
1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
3	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
4	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
5	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
6	(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

(1):$P(sum\ of\ numbers \leq 4) = \frac{6}{36} = \frac{1}{6}$

(2): $P(at\ least\ one\ of\ the\ dice\ shows\ 4) = \frac{11}{36}$

(3):$P(one\ of\ the\ dice\ shows\ 1\ and\ the\ sum\ is\ 4) = \frac{2}{36}$

(4):

$$A = \{(1,3), (2,2), (3,1)\}$$

$$B = \{(1,2), (3,2), (4,2), (5,2), (6,2), (2.1), (2,3), (2,4), (2,5), (2,6)\}$$

$A \cap B = \emptyset$, and hence, A and B are disjoint.

Solution of Q4 - p.2:

Givens:

$$P(A) = 0.3, \quad P(B) = 0.4, \quad P(A \cap B \cap C) = 0.03, \quad P(\overline{A \cap B}) = 0.88$$

(1):

$$P(A \cap B) = 1 - P(\overline{A \cap B}) = 1 - 0.88 = 0.12$$

$$P(A) \times P(B) = 0.3 \times 0.4 = 0.12$$

$$\Rightarrow P(A \cap B) = P(A) \times P(B)$$

Therefore, A and B are independent.

(2):

$$P(C \mid A \cap B) = \frac{P(A \cap B \cap C)}{P(A \cap B)} = \frac{0.03}{0.12} = 0.25$$

Solution of Q5 - p.2:

R: *will rain tomorrow*.

R^c: *will not rain tomorrow*.

$$P(R^c) = 1 - P(R)$$
$$P(R^c) = 1 - 0.23$$
$$P(R^c) = 0.77$$

Solution of Q6 - p.3:

Givens:

$$P(R) = 0.7, \quad P(J) = 0.4, \quad P(R \cup J) = 0.8$$

(1): $P(R \cup J) = P(R) + P(J) - P(R \cap J)$

$$\Rightarrow 0.8 = 0.7 + 0.4 - P(R \cap J)$$

$$P(R \cap J) = \mathbf{0.3}$$

(2): $P(\bar{R} \cap \bar{J}) = 1 - P(R \cup J) = 1 - 0.8 = \mathbf{0.2}$

Solution of Q7 - p.3:

C : *the specimen is contaminated*.

C^c: *the specimen is not contaminated*.

$$P(C) = 0.1$$
$$P(C^c) = 1 - P(C) = 1 - 0.1 = 0.9$$

(1)

$$P(C^c \cap C^c \cap C^c) = P(C^c)P(C^c)P(C^c) = 0.9 \times 0.9 \times 0.9 = 0.729$$

(2)

$$P(C \cap C^c \cap C^c) + P(C^c \cap C \cap C^c) + P(C^c \cap C^c \cap C)$$

$$= P(C)P(C^c)P(C^c) + P(C^c)P(C)P(C^c) + P(C^c)P(C^c)P(C)$$

$$= (0.1 \times 0.9 \times 0.9) + (0.9 \times 0.1 \times 0.9) + (0.9 \times 0.9 \times 0.1)$$

$$= 3 \times (0.1 \times 0.9 \times 0.9) = 0.243$$

Solution of Q8 - p.3:

	Male (M)	*Female (F)*	*total*
Elementary (E)	*28*	*50*	*78*
Secondary (S)	*38*	*45*	*83*
College (C)	*22*	*17*	*39*
total	*88*	*112*	*200*

(1):

$$P(M) = \frac{88}{200} = 0.44$$

(2):

$$P(M \mid S) = \frac{P(M \cap S)}{P(S)} = \frac{38/200}{83/200} = \frac{38}{83} = 0.46$$

(3):

$$P(C^c \mid F) = \frac{P(C^c \cap F)}{P(F)} = \frac{(50+45)/200}{112/200} = \frac{95/200}{112/200} = \frac{95}{112} = 0.85$$

(4): $we\ will\ check\ if$

$$P(M \cap E) \stackrel{?}{=} P(M) \times P(E)$$

$$\frac{28}{200} \neq \frac{88}{200} \times \frac{78}{200}$$

Thus, M and E are dependent.

Or: $we\ can\ also\ check\ if$

$$P(M) \stackrel{?}{=} P(M \mid E)$$

$$0.44 \neq \frac{28}{78}$$

Thus, M and E are dependent.

Solution of Q9 - p.3:

	Male (M)	Female (F)	total
Daily (D)	300	50	350
Occasionally (O)	200	50	250
Not at all (N)	100	300	400
total	600	400	1000

1.
$$P(F) = \frac{400}{1000} = 0.4$$

2.
$$P(F \cap D) = \frac{50}{1000} = 0.05$$

3.
$$P(F \mid D) = \frac{P(F \cap D)}{P(D)} = \frac{50/1000}{350/1000} = \frac{50}{350} = 0.1429$$

4. $we\ will\ check\ if$

$$P(F \cap D) \stackrel{?}{=} P(F) \times P(D)$$

$$\frac{50}{1000} \neq \frac{400}{1000} \times \frac{350}{1000}$$

Thus, F and D are dependent.

Or: $we\ can\ also\ check\ if$

$$P(F) \stackrel{?}{=} P(F \mid D)$$

$$0.4 \neq 0.1429$$

Solution of Q10 - p.3:

$$P(E1) = 0.4\,,\quad P(E2) = 0.6\,,\quad E1\ \&\ E2\ are\ independent$$

$$P(E1 \cap E2) = P(E1) \times P(E2) = 0.4 \times 0.6 = 0.24$$

Solution of Q11 - p.4:

$$P(B) = 0.3,\ \ P(A \mid B) = 0.4,\ \ P(A \cap B) = ?$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$\Rightarrow 0.4 = \frac{P(A \cap B)}{0.3}$$

$$\Rightarrow P(A \cap B) = 0.4 \times 0.3 = 0.12$$

Solution of Q12 - p.4:

E: electrical frailer.

V: virus.

$$P(E) = 0.15,\ \ P(V) = 0.25,\ \ P(E \cap V) = 0.10$$

$$P(E \cup V) = P(E) + P(V) - P(E \cap V)$$

$$P(E \cup V) = 0.15 + 0.25 - 0.10 = 0.3$$

Solution of Q13 - p.4:

Givens:

$$B = 4\ \ \&\ \ G = 2\ \ \&\ \ Total = 6$$

$$P(B) = \frac{4}{6} = \frac{2}{3}\ \ \ \&\ \ \ P(G) = \frac{2}{6} = \frac{1}{3}$$

The probability of getting two Green balls and one Black ball with replacement is:

$$P(GGB) + P(GBG) + P(BGG) = 3\left(\frac{1}{3} \times \frac{1}{3} \times \frac{2}{3}\right) = \frac{6}{27}$$

Solution of Q16 - p.4:

Givens:

$$P(A_1) = 0.4,\ \ P(A_1 \cap A_2) = 0.2,\ \ P(A_3 \mid A_1 \cap A_2) = 0.75$$

(1)

$$P(A_2|A_1) = \frac{P(A_1 \cap A_2)}{P(A_1)} = \frac{0.2}{0.4} = 0.5$$

(2)

$$P(A_3|\ A_1 \cap A_2) = \frac{P(A_1 \cap\ A_2 \cap A_3)}{P(A_1 \cap A_2)}$$

$$0.75 = \frac{P(A_1 \cap\ A_2 \cap A_3)}{0.2}$$

$$P(A_1 \cap\ A_2 \cap A_3) = 0.75 \times 0.2 = 0.15$$

Solution of Q17 - p.4:

Givens:

$$P(A) = 0.9,\ \ P(B) = 0.6,\ \ P(A \cap B) = 0.5$$

1. $P(A \cap B^c) = P(A) - P(A \cap B) = 0.9 - 0.5 = 0.4$

2. $P(A^c \cap B^c) = 1 - P(A \cup B)$
$= 1 - [P(A) + P(B) - P(A \cap B)]$
$= 1 - [0.9 + 0.6 - 0.5] = 0$

3.
$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{0.5}{0.9} = 0.5556$$

4. $we\ can\ see\ that\ P(A \cap B) = 0.5 \neq 0$

Thus, A and B are Joint.

5. $we\ have\ to\ check\ if:$

$$P(A \cap B) \stackrel{?}{=} P(A) \times P(B)$$
$$0.5 \neq 0.9 \times 0.6$$

A and B are dependent.

Bayes Rule:

If the events $A_1, A_2, \ldots,$ and A_n constitute a partition of the sample space S such that $P(A_i) \neq 0$ for $i = 1, 2, \ldots\, n$, then for any event B:

$$P(B) = \sum_{i=1}^{n} P(A_i)P(B \mid A_i) = \sum_{i=1}^{n} P(A_i \cap B)$$

Note:

$$P(A_i \mid B) = \frac{P(A_i)P(B \mid A_i)}{P(B)}$$

<u>Solution of Q3 - p.6:</u>

Givens:

$$M = Improperly\ sealed$$

$$P(A) = 0.5 -- \rightarrow P(M \mid A) = 0.04$$

$$P(B) = 0.3 -- \rightarrow P(M \mid B) = 0.06$$

$$P(C) = 0.2 -- \rightarrow P(M \mid C) = 0.12$$

(a): $P(B \cap M) = P(M \mid B) \times P(B) = 0.06 \times 0.3 = 0.018$

(b): $P(M) = P(A) \times P(M \mid A) + P(B) \times P(M \mid B) + P(C) \times P(M \mid C)$

$$= (0.5 \times 0.04) + (0.3 \times 0.06) + (0.2 \times 0.12) = 0.062$$

(c):

$$P(B \mid M) = \frac{P(B \cap M)}{P(M)} = \frac{P(B) \times P(M \mid B)}{P(M)} = \frac{0.3 \times 0.06}{0.062} = 0.2903$$

Solution of Q4 - p.6:

Givens:

$$B = Broken\ Dish$$

$$P(M) = 0.5 - -\rightarrow P(B \mid M) = 0.4$$

$$P(A) = 0.5 - -\rightarrow P(B \mid A) = 0.1$$

(a): $P(B) = P(M) \times P(B \mid M) + P(A) \times P(B \mid A)$

$$= (0.5 \times 0.4) + (0.5 \times 0.1) = 0.25$$

(b):

$$P(M \mid B) = \frac{P(M \cap B)}{P(B)} = \frac{P(M) \times P(B \mid M)}{P(B)} = \frac{0.5 \times 0.4}{0.25} = 0.8$$

Solution of Q5 - p.6:

Givens:

$$P = pass\ the\ program$$

$$P(A) = \frac{100}{400} = 0.25 - -\rightarrow P(P \mid A) = 0.9$$

$$P(B) = \frac{300}{400} = 0.75 - -\rightarrow P(P \mid B) = 0.7$$

1.

$$P(P) = P(A) \times P(P \mid A) + P(B) \times P(P \mid B)$$
$$= (0.25 \times 0.9) + (0.75 \times 0.7) = 0.75$$

2.

$$P(A \mid P) = \frac{P(A \cap P)}{P(P)} = \frac{P(A) \times P(P \mid A)}{P(P)} = \frac{0.25 \times 0.9}{0.75} = 0.3$$

Discrete Random Variables:

- $0 \leq f(x) \leq 1$
- $\sum f(x) = 1$
- $f(x) = P(X = x)$
- $E(X) = \sum x\, f(x)$
- $Var(X) = E(X^2) - E(X)^2$
- $E(X^2) = \sum x^2 f(x)$
- $E(aX \pm b) = aE(X) \pm b$
- $Var(aX \pm b) = a^2 Var(X)$
- $F(x) = P(X \leq x)$

Solution of Q1 – P.7:

(a): $S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$

(b): $X = \#\ of\ Head - \#\ of\ Tails \rightarrow X = -3, -1, 1, 3$

(c): We have a balanced coin, and hence, $P(H) = P(T) = {}^{1}/_{2}$

$* \; P(-3) = P(TTT) = P(T)P(T)P(T)$

$= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$

$* \; P(-1) = P(HTT) + P(THT) + P(TTH)$

$= \left(\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}\right) + \left(\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}\right) + \left(\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}\right) = \frac{3}{8}$

$* \; P(1) = P(HHT) + P(HTH) + P(THH) =$

$= \left(\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}\right) + \left(\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}\right) + \left(\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}\right) = \frac{3}{8}$

$* \; P(3) = P(HHH) = P(H)P(H)P(H)$

$= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$

x	-3	-1	1	3
$f(x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

(d): $P(X \leq 1) = 1 - P(X > 1) = 1 - f(3) = \frac{7}{8}$

(e): $P(X < 1) = f(-1) + f(-3) = \frac{4}{8} = \frac{1}{2}$

(f): $E(X) = \sum x\, f(x) = \left(-3 \times \frac{1}{8}\right) + \left(-1 \times \frac{3}{8}\right) + \left(1 \times \frac{3}{8}\right) + \left(3 \times \frac{1}{8}\right) = 0$

(g): $Var(X) = E(X^2) - E(X)^2$, we need to find $E(X^2)$:

$E(X^2) = \left(-3^2 \times \frac{1}{8}\right) + \left(-1^2 \times \frac{3}{8}\right) + \left(1^2 \times \frac{3}{8}\right) + \left(3^2 \times \frac{1}{8}\right) = \frac{24}{8} = 3$

Hence, $Var(X) = 3 - 0^2 = 3$

Solution of Q2 – P.7:

(1): $S = \{MM, MF, FM, FF\}$

(2): $X = \# \; of \; Females \rightarrow x = 0, 1, 2$

(3): We have: $P(M) = 0.8$ & $P(F) = 0.2$

$P(0) = P(MM) = P(M)P(M) = 0.8 \times 0.8 = 0.64$

$P(1) = P(MF) + P(FM) = P(M)P(F) + P(F)P(M) = (0.8 \times 0.2) + (0.2 \times 0.8) = 0.32$

$P(2) = P(FF) = P(F)P(F) = 0.2 \times 0.2 = 0.04$

x	0	1	2
$f(x)$	0.64	0.32	0.04

(4): $P(X \geq 1) = 1 - P(X < 1) = 1 - f(0) = 0.36$

(5): $P(X \leq 1) = f(0) + f(1) = 0.96$

(6): $E(X) = \sum x\, f(x) = (0 \times 0.64) + (1 \times 0.32) + (2 \times 0.04) = 0.4$

(7): $Var(X) = E(X^2) - E(X)^2$, we need to find $E(X^2)$:

$E(X^2) = (0^2 \times 0.64) + (1^2 \times 0.32) + (2^2 \times 0.04) = 0.48$

Hence, $Var(X) = E(X^2) - E(X)^2$

$= 0.48 - 0.4^2 = 0.32$

Solution of Q3 – P.7:

Givens:

$$P(5) = 0.4\,,\quad P(10) = 0.6$$

(i): $\boldsymbol{S} = \{(\mathbf{5},\mathbf{5}),(\mathbf{5},\mathbf{10}),(\mathbf{10},\mathbf{5}),(\mathbf{10},\mathbf{10})\}$

(ii): X = the total sum of the two numbers $\rightarrow X = 10, 15, 20$

(iii): $X = 10, 15, 20$

$$P(X = 10) = P(5{,}5) = 0.4 \times 0.4 = \mathbf{0.16}$$

$$P(X = 15) = P(5,10) + P(10,5) = (0.4 \times 0.6) + (0.6 \times 0.4) = \mathbf{0.48}$$

$$P(X = 20) = P(10,10) = 0.6 \times 0.6 = \mathbf{0.36}$$

Therefore,

x	10	15	20
$f(x)$	0.16	0.48	0.36

(iv): $P(X = 0) = \mathbf{0}$

(v): $P(X > 10) = f(15) + f(20) = 0.48 + 0.36 = \mathbf{0.84}$

(vi): $E(X) = \sum x\, f(x) = \mathbf{16}$

(vii): $Var(X) = E(X^2) - [E(X)]^2$

$$E(X^2) = 268, \quad Var(x) = 268 - 16^2 = \mathbf{12}$$

Solution of Q4 – P.7:

x	-3	6	9
$f(x)$	0.1	0.5	0.4

(1): $E(X) = (-3 \times 0.1) + (6 \times 0.5) + (9 \times 0.4) = 6.3$

(2): $E(X^2) = (-3^2 \times 0.1) + (6^2 \times 0.5) + (9^2 \times 0.4) = 51.3$

(3): $Var(X) = E(X^2) - [E(X)]^2 = 51.3 - 6.3^2 = 11.61$

(4): $E(2X + 1) = 2E(X) + 1 = (2 \times 6.3) + 1 = 13.6$

(5): $Var(2X + 1) = 2^2 Var(X) = 4 \times 11.61 = 46.44$

Solution of Q5 – P.8:

a.

$$f(x) = \frac{x+1}{10}; \quad x = 0, 1, 2, 3, 4$$

x	0	1	2	3	4
$f(x)$	$^1/_{10}$	$^2/_{10}$	$^3/_{10}$	$^4/_{10}$	$^5/_{10}$

$$f(x)\ is\ not\ a\ P.D.F\ because \sum f(x) \neq 1$$

b.

$$f(x) = \frac{x-1}{5}; \quad x = 0, 1, 2, 3, 4$$

x	0	1	2	3	4
$f(x)$	$^{-1}/_5$				

$$f(x)\ is\ not\ a\ P.D.F\ because\ every\ f(x)\ shoud\ be\ 0 \leq f(x) \leq 1$$

c.

$$f(x) = \frac{1}{5}; \quad x = 0, 1, 2, 3, 4$$

x	0	1	2	3	4
$f(x)$	$^1/_5$	$^1/_5$	$^1/_5$	$^1/_5$	$^1/_5$

$$f(x)\ is\ a\ P.D.F$$

d.

$$f(x) = \frac{5-x^2}{6}; \quad x = 0, 1, 2, 3$$

x	0	1	2	3
$f(x)$				$^{-4}/_6$

$$f(x)\ is\ not\ a\ P.D.F\ because\ every\ f(x)\ shoud\ be\ 0 \leq f(x) \leq 1$$

Solution of Q6 – P.8:

$$f(x) = P(X = x) = \frac{1}{3}; \quad x = 0, 1, 2$$

x	0	1	2
$f(x)$	$1/3$	$1/3$	$1/3$

1.

$$E(X) = \sum x\, f(x) = \left(0 \times 1/3\right) + \left(1 \times 1/3\right) + \left(2 \times 1/3\right) = 1$$

2.

$$Var(X) = E(X^2) - E(X)^2$$

$$E(X^2) = \sum X^2\, f(x) = \left(0^2 \times 1/3\right) + \left(1^2 \times 1/3\right) + \left(2^2 \times 1/3\right) = 1.67$$

$$Var(X) = E(X^2) - E(X)^2$$

$$Var(X) = 1.67 - 1^2 = 0.67$$

Solution of Q7 – P.8:

$$f(x) = kx\, ; x = 1,2,3$$

x	1	2	3
$f(x)$	*k*	*2k*	*3k*

(i):

$$\sum f(x) = 1 \Rightarrow k + 2k + 3k = 1 \Rightarrow \boldsymbol{k} = \mathbf{\frac{1}{6}}$$

x	1	2	3
$f(x)$	$1/6$	$2/6$	$3/6$

(ii):

$$F(x) = \begin{cases} 0 & x < 1 \\ 1/6 & 1 \le x < 2 \\ 3/6 & 2 \le x < 3 \\ 1 & x \ge 3 \end{cases}$$

(iii): $P(0.5 < X \le 2.5) = P(X \le 2.5) - P(X \le 0.5)$

$$= F(2.5) - F(0.5) = \frac{3}{6} - 0 = \mathbf{\frac{3}{6}}$$

Solution of Q8 – P.8:

(a):

$P(2) = F(2) - F(1) = 1 - 0.6 = 0.4$

$P(1) = F(1) - F(0) = 0.6 - 0.25 = 0.35$

$P(0) = F(0) = 0.25$

$$F(x) = \begin{cases} 0 & ; \quad x < 0 \\ 0.25 & ; 0 \le x < 1 \\ 0.6 & ; 1 \le x < 2 \\ 1 & ; \quad x \ge 2 \end{cases}$$

x	0	1	2
$f(x)$	0.25	0.35	0.4

(b): $P(1 \le X < 2) =$

By using $f(x)$: $f(1) = 0.35$

By using $F(x)$: $P(X \le 1) - P(X \le 0) = F(1) - F(0) = 0.6 - 0.25 = 0.35$

(c): $P(X > 2) = 1 - P(X \le 2)$

By using $f(x)$: $P(X > 2) = 0$

By using $F(x)$: $1 - F(2) = 1 - 1 = 0$

Solution of Q11 – P.8:

Givens: $f(x) = k\binom{2}{x}\binom{3}{3-x}; \quad x = 0,1,2$

$$\sum f(x) = 1 \ \rightarrow k + 6k + 3k = 1 \Rightarrow 10k = 1 \Rightarrow k = \mathbf{0.1}$$

Solution of Q12 – P.8:

x	0	1	2	3	4
f(x)	$1/16$	$4/16$	$6/16$	$4/16$	$1/16$

(a): $P(X = 2) = \frac{11}{16} - \frac{5}{16} = \frac{6}{16} = \mathbf{\frac{3}{8}}$

(b): $P(2 \le X < 4) = P(X < 4) - P(X < 2) = \frac{15}{16} - \frac{5}{16} = \mathbf{\frac{10}{16}}$

Continuous Random Variables:

- $0 \leq f(x) \leq 1$
- $\int_{-\infty}^{\infty} f(x)dx = 1$
- $P(a < X < b) = \int_a^b f(x)dx$
- $E(X) = \int_{-\infty}^{\infty} x\, f(x)dx$
- $Var(X) = E(X^2) - E(X)^2$
- $E(X^2) = \int_{-\infty}^{\infty} x^2\, f(x)dx$
- $E(aX \pm b) = aE(X) \pm b$
- $Var(aX \pm b) = a^2 Var(X)$
- $F(x) = P(X \leq x)$

Solution of Q1 – P.10:

Givens:

$$x \text{ is a continuous random variable.}$$

$$\mu = 16 \quad \& \quad \sigma^2 = 5$$

$$P(X = 16) = 0$$

Note, $for\ any\ continuous\ random\ variable, P(x = a) = 0$

Solution of Q2 – P.10:

Givens:

$$f(x) = k\sqrt{x} \quad , \quad 0 < x < 1$$

(1): $\int_{-\infty}^{\infty} f(x)dx = 1 \Rightarrow \int_0^1 k\sqrt{x}\, dx = 1$

$$\int_0^1 k\sqrt{x}\, dx = 1 \Rightarrow \int_0^1 kx^{0.5}\, dx = 1 \Rightarrow k\left[\frac{x^{1.5}}{1.5}\right]_0^1 = 1 \Rightarrow \frac{k}{1.5} = 1 \Rightarrow \boldsymbol{k = 1.5}$$

$$then, \qquad f(x) = 1.5\sqrt{x} \quad , \quad 0 < x < 1$$

(2): $P(0.3 < X < 0.6) = \int_{0.3}^{0.6} f(x)dx$

$$\int_{0.3}^{0.6} 1.5x^{0.5}dx = 1.5\left[\frac{x^{1.5}}{1.5}\right]_{0.3}^{0.6} = [x^{1.5}]_{0.3}^{0.6} = 0.6^{1.5} - 0.3^{1.5} = \boldsymbol{0.3004}$$

(3): $E(X) = \int_{-\infty}^{\infty} x\, f(x)dx$

$$E(X) = \int_{-\infty}^{\infty} x\, f(x)dx = \int_0^1 1.5x^{1.5}dx = 1.5\left[\frac{x^{2.5}}{2.5}\right]_0^1 = \frac{1.5}{2.5}[x^{2.5}]_0^1 = \boldsymbol{0.6}$$

Solution of Q3 – P.10:

Givens:

$$f(x) = k(x+1) \quad , \quad 0 < x < 2$$

(1): $\int_{-\infty}^{\infty} f(x)dx = 1 \Rightarrow \int_0^2 k(x+1)\, dx = 1$

$$\int_0^2 k(x+1)\, dx = 1 \Rightarrow k\left[\frac{x^2}{2} + x\right]_0^2 = 1 \Rightarrow 4k = 1 \Rightarrow \boldsymbol{k} = \mathbf{\frac{1}{4}}$$

$$then, \;\; f(x) = \frac{1}{4}(x+1) \quad , \quad 0 < x < 2$$

(2): $P(0 < X < 1) = \int_0^1 f(x)dx$

$$\int_0^1 \frac{1}{4}(x+1)\, dx = \frac{1}{4}\left[\frac{x^2}{2} + x\right]_0^1 = \frac{3}{8}$$

(3): $F(x) = \int_{-\infty}^{x} f(x)dx = \int_0^x f(x)dx$

$$= \int_0^x \frac{1}{4}(x+1)dx = \frac{1}{4}\left[\frac{x^2}{2} + x\right]_0^x = \frac{1}{4}\left(\frac{x^2}{2} + x\right)$$

$$then, \qquad F(x) = \frac{1}{4}\left(\frac{x^2}{2} + x\right) ; \quad 0 < x < 2$$

(4): By using $F(x)$,

$$P(0 < X < 1) = P(X < 1) - P(X < 0)$$

$$F(1) - F(0) = \frac{3}{8} - 0 = \frac{3}{8}$$

Solution of Q4 – P.10:

Givens:

$$f(x) = \frac{3x^2}{2} \qquad ; \ -1 < x < 1$$

(1): $P(0 < X < 1) = \int_0^1 f(x)dx$

$$\int_0^1 \frac{3x^2}{2}\, dx = \frac{3}{2}\left[\frac{x^3}{3}\right]\begin{matrix}1\\0\end{matrix} = \frac{1}{2}$$

(2): $E(X) = \int_{-\infty}^{\infty} x\, f(x)dx$

$$E(X) = \int_{-\infty}^{\infty} x\, f(x)dx = \int_{-1}^{1} \frac{3x^3}{2} dx = \frac{3}{2}\left[\frac{x^4}{4}\right]\begin{matrix}1\\-1\end{matrix} = 0$$

(3): $Var(X) = E(X^2) - E(X)^2$

$$E(X^2) = \int_{-\infty}^{\infty} x^2\, f(x)dx = \int_{-1}^{1} \frac{3x^4}{2} dx = \frac{3}{2}\left[\frac{x^5}{5}\right]\begin{matrix}1\\-1\end{matrix} = \frac{3}{5}$$

$$Var(X) = E(X^2) - E(X)^2 = \frac{3}{5} - 0^2 = \frac{3}{5}$$

(4): $E(2X + 3) = 2E(X) + 3 = 2 \times 0 + 3 = 3$

(5): $Var(2X + 3) = 4Var(X) = 4 \times \frac{3}{5} = \frac{12}{5}$

Solution of Q9 – P.11:

$$P(X \le x) = F(x) = \begin{cases} 0 & ; x < 0 \\ \dfrac{x}{x+1} & ; x \ge 0 \end{cases}$$

1. $P(0 < X < 2) = P(X < 2) - P(X < 0) = F(2) - F(0) = \frac{2}{3} - 0 = 0.67$
2. $P(X < k) = F(k) = 0.5$

$$\Rightarrow \frac{k}{k+1} = 0.5 \Rightarrow k = 1$$

Chebyshev's Theorem:

$$P(\mu - k\sigma < X < \mu + k\sigma) \geq 1 - \frac{1}{k^2}$$

$$or$$

$$P(|X - \mu| < k\sigma) \geq 1 - \frac{1}{k^2}$$

Solution of Q1- P.12:

Givens:

$$P(\mu - 2\sigma < X < \mu + 2\sigma)$$

By comparison with Chebyshev's Theorem, we can find that:

$$k = 2 \rightarrow 1 - \frac{1}{k^2}$$

$$= 1 - \frac{1}{2^2} = 1 - \frac{1}{4} = 0.75$$

Solution of Q2- P.11:

Givens:

$$P(3 < X < 21) \quad , \quad \mu = 12, \quad \sigma^2 = 9$$

$$P(\mu - k\sigma < X < \mu + k\sigma)$$

By taking the lower or the upper value of the given probability:

$$\mu + k\sigma = 21 \Rightarrow 12 + 3k = 21 \Rightarrow k = 3$$

Or $\mu - k\sigma = 3 \Rightarrow 12 - 3k = 3 \Rightarrow k = 3$

$$k = 3 \Rightarrow 1 - \frac{1}{k^2}$$

$$= 1 - \frac{1}{3^2} = 1 - \frac{1}{9} = \frac{8}{9}$$

Solution of Q3- P.12:

(i): Givens:

$$\begin{matrix} P(1 < X < 9) \\ P(\mu - k\sigma < X < \mu + k\sigma) \end{matrix} \quad , \quad \mu = 5 \quad \& \quad \sigma^2 = 4$$

By taking the lower or the upper value of the given probability:

$$\mu + k\sigma = 9 \Rightarrow 5 + 2k = 9 \Rightarrow k = 2$$

Or $$\mu - k\sigma = 1 \Rightarrow 5 - 2k = 1 \Rightarrow k = 2$$

$$k = 2 \Rightarrow 1 - \frac{1}{k^2} = 1 - \frac{1}{2^2} = 1 - \frac{1}{4} = 0.75$$

(ii): Givens:

$$P(a < X < b) \approx \frac{15}{16}, \quad \mu = 5, \quad \sigma^2 = 4$$

To find the constants, a & b, we have to find the value *k*:

$$1 - \frac{1}{k^2} = \frac{15}{16}$$

$$\Rightarrow 1 - \frac{15}{16} = \frac{1}{k^2}$$

$$\Rightarrow \frac{1}{k^2} = \frac{1}{16}$$

$$\Rightarrow k^2 = 16 \Rightarrow k = 4$$

$$a = \mu - k\sigma = 5 - 4 \times 2 = -3$$

$$b = \mu + k\sigma = 5 + 4 \times 2 = 13$$

$$P(-3 < X < 13) \approx \frac{15}{16}$$

Solution of Q4- P.12:

Givens:

$$f(x) = \frac{1}{10} \;,\; 0 < x < 10$$

$$\mu = 5 \;\; \& \;\; \sigma = 2.89$$

1. *Exact value of* $P(\mu - 1.5\sigma < X < \mu + 1.5\sigma)$:

$$P(\mu - 1.5\sigma < X < \mu + 1.5\sigma) = P(5 - (1.5 \times 2.89) < X < 5 + (1.5 \times 2.89))$$

$$P(0.665 < X < 9.335) = \int_{0.665}^{9.335} \frac{1}{10}\, dx = 0.867$$

2. *Using Chebyshev's theorem:*

$$P(\mu - 1.5\sigma < X < \mu + 1.5\sigma) \geq 1 - \frac{1}{k^2}$$

$$k = 1.5 \Rightarrow 1 - \frac{1}{k^2}$$

$$= 1 - \frac{1}{1.5^2} = 0.55$$

Solution of Q5- P.12:

Givens:

$$E(X) = 30 \,, Var(X) = 4$$
$$E(Y) = 10 \,, Var(Y) = 2$$

1.

$$E(2X - 3Y - 10) = 2E(X) - 3E(Y) - 10$$
$$= (2 \times 30) - (3 \times 10) - 10 = 20$$

2.

$$Var(2X - 3Y - 10) = 4Var(X) + 9Var(Y)$$
$$(4 \times 4) + (9 \times 2) = 34$$

3.

Similar to Q2 and Q3.

FOR 1st MIDTERM

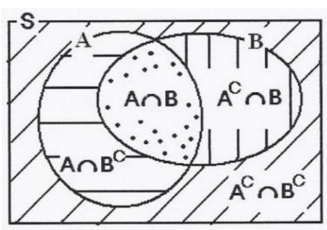
Introduction:

- $\bar{X} = \frac{\sum_{i=1}^{n} x_i}{n}$
- $S^2 = \frac{\sum_{i=1}^{n}(x_i - \bar{X})^2}{n-1}$
- $S^2 = \frac{\sum_{i=1}^{n} x_i - n\bar{X}^2}{n-1}$

Combinations &Permutations:

- $nCr = \binom{n}{r} = \frac{n!}{r!(n-r)!}$
- $nPr = \frac{n!}{(n-r)!}$

Probability:



6- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

7- $P(A|B) = P(A \cap B)/P(B)$

8- $P(A \cap B) = P(A) \times P(B)$

$(if\ A\ \&\ B\ are\ independent.)$

9- $P(A \cap B) = 0$

$(if\ A\ \&\ B\ are\ disjoint.)$

10- $P(A^c) = 1 - P(A)$

Bayes Rule:

$$P(B) = \sum_{i=1}^{n} P(A_i)P(B \mid A_i)$$

$$P(A_i \mid B) = \frac{P(A_i)P(B \mid A_i)}{P(B)}$$

Discrete Random Variables:

- $\leq f(x) \leq 1$
- $\sum f(x) = 1$
- $f(x) = P(X = x)$
- $E(X) = \sum x\, f(x)$
- $Var(X) = E(X^2) - E(X)^2$
- $E(X^2) = \sum x^2 f(x)$
- $E(aX \pm b) = aE(X) \pm b$
- $Var(aX \pm b) = a^2 Var(X)$
- $F(x) = P(X \leq x)$

Continuous Random Variables:

- $0 \leq f(x) \leq 1$
- $\int_{-\infty}^{\infty} f(x)dx = 1$
- $P(a < X < b) = \int_a^b f(x)dx$
- $E(X) = \int_{-\infty}^{\infty} x\, f(x)dx$
- $Var(X) = E(X^2) - E(X)^2$
- $E(X^2) = \int_{-\infty}^{\infty} x^2\, f(x)dx$
- $E(aX \pm b) = aE(X) \pm b$
- $Var(aX \pm b) = a^2 Var(X)$
- $F(x) = P(X \leq x)$

Chebyshev's Theorem:

$$P(\mu - k\sigma < X < \mu + k\sigma) \geq 1 - \frac{1}{k^2}$$

$$or$$

$$P(|X - \mu| < k\sigma) \geq 1 - \frac{1}{k^2}$$

Discrete Uniform:

$$f(x) = {}^{1}/_{k}\ ; \quad x = x_1, x_2, \cdots, x_k$$

Solution of Q1 – P.14:

X have discrete uniform with parameter k = 3, x = 0, 1, 2.

x	0	1	2
$f(x)$	$^{1}/_{3}$	$^{1}/_{3}$	$^{1}/_{3}$

a. $P(X = 1) = \ ^{1}/_{3}$

b. $E(X) = \left(0 \times \ ^{1}/_{3}\right) + \left(1 \times \ ^{1}/_{3}\right) + \left(2 \times \ ^{1}/_{3}\right) = 1$

c. $Var(X) = \ E(X^2) - E(X)^2$

$$E(X^2) = \left(0^2 \times \ ^{1}/_{3}\right) + \left(1^2 \times \ ^{1}/_{3}\right) + \left(2^2 \times \ ^{1}/_{3}\right) = {}^{5}/_{3}$$

$$Var(X) = \ E(X^2) - E(X)^2$$
$$Var(X) = {}^{5}/_{3} - 1 = {}^{2}/_{3}$$

Binomial Distribution:

$$f(x) = \binom{n}{x} p^x q^{n-x} \quad ; \quad x = 0,1 \dots, n$$

$$* E(X) = np \qquad * Var(X) = npq$$

$$q = 1 - p$$

Solution of Q1 – P.14:

$$p = \frac{4}{12} = \frac{1}{3} \quad , \quad n = 3$$

(a):

$$f(x) = \binom{3}{x} \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{3-x} \quad ; \quad x = 0,1,2,3$$

(b):

(i): $P(X = 0) = f(0) = \binom{3}{0} \left(\frac{1}{3}\right)^0 \left(\frac{2}{3}\right)^3 = 0.296$

(ii): $P(X = 1) = f(1) = \binom{3}{1} \left(\frac{1}{3}\right)^1 \left(\frac{2}{3}\right)^2 = 0.44$

(iii): $P(X \geq 1) = 1 - P(X < 1) = 1 - f(0) = 1 - 0.296 = 0.704$

(c):

$$E(X) = np = 3 \times \frac{1}{3} = 1$$

(d):

$$Var(X) = npq = 3 \times \frac{1}{3} \times \frac{2}{3} = \frac{2}{3}$$

Solution of Q3 – P.14:

$$p = 0.4 \quad , \quad n = 10$$

(1): $E(X) = np = 10 \times 0.4 = 4$

(2): $Var(X) = npq = 10 \times 0.4 \times 0.6 = 2.4$

(3)

$$f(x) = \binom{10}{x}(0.4)^x(0.6)^{10-x} \quad ; \quad x = 0,1, \ldots . ,10$$

$$P(X = 4) = f(4) = \binom{10}{4}(0.4)^4(0.6)^6 = 0.251$$

(4):

$$P(X < 3) = f(0) + f(1) + f(2)$$

$$= \binom{10}{0}(0.4)^0(0.6)^{10} + \binom{10}{1}(0.4)^1(0.6)^9 + \binom{10}{2}(0.4)^2(0.6)^8 = 0.167$$

(5):

$$P(X > 8) = f(9) + f(10)$$

$$= \binom{10}{9}(0.4)^9(0.6)^1 + \binom{10}{10}(0.4)^{10}(0.6)^0 = 0.0017$$

Solution of Q9 – P.15:

$$X \sim Binomial(n, p) \;\; \& \;\; E(X) = 1 \;\; \& \;\; Var(X) = 0.75$$

$$\frac{Var(X)}{E(X)} = \frac{0.75}{1} \Rightarrow \frac{npq}{np} = \frac{0.75}{1} \Rightarrow q = 0.75 \Rightarrow p = 0.25.$$

$$E(X) = 1 \Rightarrow np = 1 \Rightarrow n \times 0.25 = 1 \Rightarrow n = 4.$$

$$f(x) = \binom{4}{x}(0.25)^x(0.75)^{4-x} \;\; ; \quad x = 0,1,2,3,4$$

$$P(X = 1) = f(1) = \binom{4}{1}(0.25)^1(0.75)^3 = 0.422$$

Solution of Q11 – P.15:

$$p = 0.75 \;\; , \;\; n = 5$$

$$f(x) = \binom{5}{x}(0.75)^x(0.25)^{5-x} \;\; ; \quad x = 0,1,2,3,4,5$$

1.

$$P(X = 0) = f(0) = \binom{5}{0}(0.75)^0(0.25)^5 = 0.00098$$

2.

$$P(X = 4) = f(4) = \binom{5}{4}(0.75)^4(0.25)^1 = 0.3955$$

3.

$$P(X \geq 4) = f(4) + f(5)$$

$$\binom{5}{4}(0.75)^4(0.25)^1 + \binom{5}{5}(0.75)^5(0.25)^0 = 0.6328$$

4.

$$E(X) = np = 5 \times 0.75 = 3.75$$

Hyper geometric Distribution:

$$f(x) = \frac{\binom{k}{x}\binom{N-k}{n-x}}{\binom{N}{n}} \quad ; \quad x = 0,1,\dots.\min(n,k)$$

$$* \; E(X) = n \times \frac{k}{N} \qquad * \; Var(X) = n \times \frac{k}{N}\left(1 - \frac{k}{N}\right)\left(\frac{N-n}{N-1}\right)$$

Solution of Q1 – P.16:

$$N = 7 \;\;,\;\; n = 3 \;\;,\;\; k = 2$$

(i):

$$f(x) = \frac{\binom{2}{x}\binom{5}{3-x}}{\binom{7}{3}} \quad ; \quad x = 0,1,2$$

(ii):

$$P(X = 0) = f(0) = \frac{\binom{2}{0}\binom{5}{3}}{\binom{7}{3}} = 0.29$$

(iii):

$$E(X) = n \times \frac{k}{N} = 3 \times \frac{2}{7} = \frac{6}{7}$$

(iv):

$$Var(X) = n \times \frac{k}{N}\left(1 - \frac{k}{N}\right)\left(\frac{N-n}{N-1}\right) = 3 \times \frac{2}{7}\left(\frac{5}{7}\right)\left(\frac{7-3}{7-1}\right) = 0.41$$

Solution of Q2 – P.16:

$$N = 5 \quad , \quad n = 2 \quad , \quad k = 3$$

$$f(x) = \frac{\binom{3}{x}\binom{2}{2-x}}{\binom{5}{2}} \quad ; \quad x = 0,1,2$$

a.

$$P(X = 0) = f(0) = \frac{\binom{3}{0}\binom{2}{2}}{\binom{5}{2}} = 0.1$$

b.

$$P(X \leq 1) = f(0) + f(1) = \frac{\binom{3}{0}\binom{2}{2}}{\binom{5}{2}} + \frac{\binom{3}{1}\binom{2}{1}}{\binom{5}{2}} = 0.7$$

c.

$$E(X) = n \times \frac{k}{N} = 2 \times \frac{3}{5} = \frac{6}{5}$$

d.

$$Var(X) = n \times \frac{k}{N}\left(1 - \frac{k}{N}\right)\left(\frac{N-n}{N-1}\right) = 2 \times \frac{3}{5}\left(1 - \frac{3}{5}\right)\left(\frac{5-2}{5-1}\right) = 0.36$$

Solution of Q3 – P.16:

$$N = 10 \quad , \quad n = 4 \quad , \quad k = 2$$

(i):

$$f(x) = \frac{\binom{2}{x}\binom{8}{4-x}}{\binom{10}{4}} \quad ; \quad x = 0,1,2$$

(ii):

$$P(X = 0) = f(0) = \frac{\binom{2}{0}\binom{8}{4}}{\binom{10}{4}} = 0.33$$

(iii):

$$P(X \geq 1) = 1 - P(X < 1) = 1 - f(0) = 0.67$$

(iv):

$$E(X) = n \times \frac{k}{N} = 4 \times \frac{2}{10} = 0.8$$

(v):

$$Var(X) = n \times \frac{k}{N}\left(1 - \frac{k}{N}\right)\left(\frac{N-n}{N-1}\right) = 4 \times \frac{2}{10}\left(1 - \frac{2}{10}\right)\left(\frac{10-4}{10-1}\right) = 0.43$$

<u>Solution of Q9 – P.17:</u>

$$N = 20 \quad , \quad n = 10 \quad , \quad k = 5$$

$$f(x) = \frac{\binom{5}{x}\binom{15}{10-x}}{\binom{20}{10}} \quad ; \quad x = 0,1,2,3,4,5$$

1.

$$P(X = 2) = f(2) = \frac{\binom{5}{2}\binom{15}{8}}{\binom{20}{10}} = 0.35$$

2.

$$P(X \leq 1) = f(0) + f(1) = \frac{\binom{5}{0}\binom{15}{10}}{\binom{20}{10}} + \frac{\binom{5}{1}\binom{15}{9}}{\binom{20}{10}} = 0.15$$

3.

$$E(X) = n \times \frac{k}{N} = 10 \times \frac{5}{20} = \frac{5}{2}$$

4.

$$Var(X) = n \times \frac{k}{N}\left(1 - \frac{k}{N}\right)\left(\frac{N-n}{N-1}\right) = 10 \times \frac{5}{20}\left(\frac{15}{20}\right)\left(\frac{20-10}{20-1}\right) = 0.99$$

Poisson distribution:

$$f(x) = \frac{e^{-\lambda}\ \lambda^{x}}{x!} \quad ; \quad x = 0,1,2,\dots$$

$$E(X) = Var(X) = \lambda$$

<u>Solution of Q1 – P.18:</u>

(i):

$$\lambda_{one\ day} = 3$$

$$f(x) = \frac{e^{-3}(3)^{x}}{x!} \ ; \quad x = 0,1,2,\dots$$

(1):

$$P(X = 0) = f(0) = \frac{e^{-3}(3)^{0}}{0!} = 0.05$$

(2):

$$P(X > 3) = 1 - P(X \le 3) = 1 - [f(0) + f(1) + f(2) + f(3)]$$

$$= 1 - [\frac{e^{-3}(3)^{0}}{0!} + \frac{e^{-3}(3)^{1}}{1!} + \frac{e^{-3}(3)^{2}}{2!} + \frac{e^{-3}(3)^{3}}{3!}] = 0.35$$

(3):

$$\lambda_{two\ days} = 6$$

$$f(x) = \frac{e^{-6}(6)^{x}}{x!} \quad ; \quad x = 0, \dots\dots \infty$$

$$P(X = 5) = f(5) = \frac{e^{-6}(6)^{5}}{5!} = 0.16$$

(ii):

$$E(X) = \lambda_{4\ days} = 4 \times 3 = 12$$

Solution of Q3 – P.18:

(a):

$$\lambda_{\,one\ day} = 4$$

$$f(x) = \frac{e^{-4}(4)^x}{x!} \quad ; \quad x = 0,1,2,\dots$$

$$P(X = 2) = f(2) = \frac{e^{-4}(4)^2}{2!} = 0.15$$

(b):

$$E(X) = \lambda_{week} = 4 \times 7 = 28$$

(c):

$$\lambda_{\,12hours} = 2$$

$$f(x) = \frac{e^{-2}(2)^x}{x!} \quad ; \quad x = 0,1,2,\dots$$

$$P(X \geq 2) = 1 - P(X < 2) = 1 - [f(0) + f(1)]$$

$$= 1 - \left[\frac{e^{-2}(2)^0}{0!} + \frac{e^{-2}(2)^1}{1!}\right] = 0.59$$

Solution of Q6 – P.18:

When we have a Binomial distribution with a small p and a large n, we can make an approximation to Poisson distribution:

$$p = 0.002 \ \ \& \ \ n = 1000 \ \ \& \ \ \lambda = np = 2$$

$$P(X = 3) = f(3) = \frac{e^{-2}2^3}{3!} = 0.18045$$

The Uniform Distribution:

$$f(x) = \frac{1}{b-a} \quad ; \quad a < x < b$$

$$* \; E(X) = \frac{a+b}{2} \qquad * \; Var(X) = \frac{(b-a)^2}{12} \qquad * \; F(x) = \frac{x-a}{b-a} \quad ; \quad a < x < b$$

Solution of Q1 – P.20:

$$f(x) = \frac{1}{10} \quad ; \; 0 < x < 10$$

(1):

$$P(X < 6) = F(6) = \frac{x-a}{b-a} = \frac{6-0}{10-0} = \frac{6}{10}$$

(2):

$$E(X) = \frac{a+b}{2} = \frac{0+10}{2} = \frac{10}{2} = 5$$

(3):

$$Var(X) = \frac{(b-a)^2}{12} = \frac{(10-0)^2}{12} = \frac{100}{12} = 8.33$$

Solution of Q2 – P.20:

$$f(x) = 3 \quad ; \quad \frac{2}{3} < x < 1$$

1. $P(0.33 < X < 0.5) = 0$
2. $P(X > 1.25) = 0$
3.

$$Var(X) = \frac{(b-a)^2}{12} = \frac{\left(1 - {}^{2}/_{3}\right)^2}{12} = 0.00926$$

Solution of Q3 – P.20:

$$f(x) = 0.2 \quad ; \quad 0 < x < 5$$

(1):

$$P(X > 1) = 1 - P(X < 1) = 1 - F(1) = 1 - \left(\frac{1-0}{5-0}\right) = 1 - \frac{1}{5} = 0.8$$

(2):

$P(X \geq 1) = 0.8$, the same as part (1) because we had a continuous random variable, and hence, $P(X = 1) = 0$

(3):

$$E(X) = \frac{a+b}{2} = \frac{0+5}{2} = 2.5$$

(4):

$$Var(X) = E(X^2) - [E(X)]^2 \Rightarrow E(X^2) = Var(X) + [E(X)]^2$$

$$Var(X) = \frac{(b-a)^2}{12} = \frac{(5-0)^2}{12} = \frac{25}{12} = 2.0833$$

$$E(X^2) = Var(X) + [E(X)]^2 = 2.08 + 2.5^2 = 8.33$$

(5):

$$Var(X) = \frac{(b-a)^2}{12} = \frac{(5-0)^2}{12} = \frac{25}{12} = 2.0833$$

(6):

$$F(x) = \frac{x-a}{b-a} \quad ; \quad a < x < b$$

$$F(x) = \frac{x}{5} \quad ; \quad 0 < x < 5$$

$$F(1) = \frac{1}{5}$$

The Normal Distribution:

Solution of Q1 – P.21:

(A):

(1): $P(Z < 1.43) = 0.9236$

(2): $P(Z < 1.39) = 0.9177$

(3): $P(Z > -0.89) = 1 - P(Z < -0.89) = 1 - 0.1867 = 0.8133$

(4):

$$P(-2.16 < Z < -0.65) = P(Z < -0.65) - P(Z < -2.16)$$

$$= 0.2578 - 0.0154 = 0.2424$$

(5):

$$P(0.93 < Z < k) = 0.0427$$

$$P(Z < k) - P(Z < 0.93) = 0.0427$$

$$\Rightarrow P(Z < k) - 0.8238 = 0.0427$$

$P(Z < k) = 0.8665 \Rightarrow k = 1.11$ By searching inside the Z table.

Solution of Q2 – P.21:

$$X \sim N(12, 0.03)$$

(1):

$$P(X < 12.05) = P\left(Z < \frac{12.05 - \mu}{\sigma}\right)$$

$$= P\left(Z < \frac{12.05 - 12}{0.03}\right) = P(Z < 1.67) = 0.9525$$

(2):

$$P(X > 11.97) = P\left(Z > \frac{11.97 - \mu}{\sigma}\right) = P\left(Z > \frac{11.97 - 12}{0.03}\right)$$

$$= P(Z > -1) = 1 - P(Z < -1) = 1 - 0.1587 = 0.8413$$

(3):

$$P(11.95 < X < 12.05) = P\left(\frac{11.95 - 12}{0.03} < Z < \frac{12.05 - 12}{0.03}\right)$$

$$= P(-1.67 < Z < 1.67) = P(Z < 1.67) - P(Z < -1.67)$$

$$0.9525 - 0.0475 = 0.905$$

<u>Solution of Q6 – P.22:</u>

$$X \sim N(128,9)$$

(1):

$$P(X \leq 110) = P\left(Z < \frac{110 - 128}{9}\right) = P(Z < -2) = 0.0228$$

(2):

$$P(X > 149) = P\left(Z > \frac{149 - 128}{9}\right) = 1 - P(Z < 2.33) = 1 - 0.9901 = 0.0099$$

(3):

$$P(X > x) = 0.86 \Rightarrow P(X < x) = 0.14 \Rightarrow P\left(Z < \frac{x - 128}{9}\right) = 0.14$$

by searching inside the table for 0.14, and transforming X to Z, we got:

$$\frac{x - 128}{9} = -1.08 \Rightarrow x = 118.28$$

(4):

$P(X < x) = 0.5$, by searching inside the table for 0.5, and transforming X to Z

$$\frac{x - 128}{9} = 0 \Rightarrow x = 128$$

Solution of Q8 – P.22:

$$P(X < \mu + 2\sigma) = P\left(Z < \frac{(\mu + 2\sigma) - \mu}{\sigma}\right) = P(Z < 2) = 0.9772$$

Solution of Q9 – P.22:

$$Given\ that\ \ \sigma = 1$$

$$P(X < 3) = 0.877 \Rightarrow P\left(Z < \frac{3 - \mu}{1}\right) = 0.877$$

$$3 - \mu = 1.16 \Rightarrow \mu = 1.84$$

Solution of Q10 – P.22:

$$X \sim N(70,5)$$

$$P(X < x) = 0.33 \Rightarrow P\left(Z < \frac{x - 70}{5}\right) = 0.33$$

by searching inside the table for 0.33, and transforming X to Z, we got:

$$\frac{x - 70}{5} = -0.44 \Rightarrow x = 67.8$$

The Exponential Distribution:

$$f(x) = \frac{1}{\beta} e^{-\frac{x}{\beta}} \quad ; \quad x > 0$$

$$E(X) = \beta \quad ; \quad Var(X) = \beta^2 \quad ; \quad F(x) = 1 - e^{-\frac{x}{\beta}} \; ; \; x > 0$$

Solution of Q1 – P.24:

$$X \sim expo(4) \quad , \quad f(x) = \frac{1}{4} e^{-\frac{x}{4}} \quad ; \quad x > 0$$

(1):

$$P(X < 8) = F(8) = 1 - e^{-\frac{8}{4}} = 1 - e^{-2} = 0.8647$$

(2):

$$Var(X) = \beta^2 = 4^2 = 16$$

Solution of Q3 – P.24:

$$f(x) = \frac{1}{200} e^{-\frac{x}{200}} \quad ; \quad x > 0$$

(1):

$$E(X) = \beta = 200$$

(2):

$$P(X > 100) = 1 - P(X < 100) = 1 - F(100)$$

$$1 - \left[1 - e^{-\frac{100}{200}}\right] = 1 - 1 + e^{-0.5} = e^{-0.5} = 0.6065$$

(3):

$P(X = 200) = 0$, because we had a continuous random variable, and hence, $P(X = a) = 0$

Solution of Q4 – P.24:

$$\beta = 6 \ , \ n = 4$$

(1):

$$Var(T) = \beta^2 = 36$$

(2):

In this part, we need the probability $P(X \leq 1)$, by using the Binomial distribution. We have $p = P(T > 6)$ & $n = 4$:

$$P(T > 6) = 1 - P(T < 6) = 1 - \left(1 - e^{-\frac{6}{6}}\right) = e^{-\frac{6}{6}} = 0.37 = p$$

Therefore, $p = 0.37$ & $n = 4$, the probability $P(x \leq 1)$ by using the Binomial distribution:

$$P(X \leq 1) = f(0) + f(1)$$

$$= \binom{4}{0}(0.37)^0(0.63)^4 + \binom{4}{1}(0.37)^1(0.63)^3 = 0.531$$

(3):

Again, we should get $P(X \geq 2)$, by using the Binomial distribution:

$$P(X \geq 2) = 1 - P(X < 2) = 1 - P(X \leq 1) = 1 - 0.531 = 0.469$$

(4):

$$E(X) = np = 4 \times 0.37 = 1.48$$

FOR 2nd MIDTERM

	$f(x)$ or PDF		$E(X)$	$Var(X)$	$F(x)$ or CDF
Discrete Uniform	$1/k$	$x = x_1, x_2, \cdots, x_k$	$\sum x\, f(x)$	$E(X^2) - E(X)^2$	-
Binomial	$\binom{n}{x} p^x q^{n-x}$	$x = 0, 1, 2, \ldots., n$	np	npq	-
Hyper geometric	$\frac{\binom{k}{x}\binom{N-k}{n-x}}{\binom{N}{n}}$	$x = 0,1, \ldots.\min(n, k)$	$n \times \frac{k}{N}$	$n \times \frac{k}{N}\left(1 - \frac{k}{N}\right)\left(\frac{N-n}{N-1}\right)$	-
Poisson	$\frac{\lambda^x e^{-\lambda}}{x!}$	$x = 0, 1, 2, \ldots.$	λ	λ	-
Continuous Uniform	$\frac{1}{b-a}$	$a < X < b$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	$\frac{x-a}{b-a}$
Exponential	$\frac{1}{\beta} e^{-\frac{x}{\beta}}$	$X > 0$	β	β^2	$1 - e^{-\frac{x}{\beta}}$

Sampling Distribution

Sampling Distribution: Single Mean

$$*\ \bar{X} \sim N\left(\mu, \frac{\sigma}{\sqrt{n}}\right)$$

$$*\ E(\bar{X}) = \bar{X} = \mu \qquad\qquad *\ Var(\bar{X}) = \frac{\sigma^2}{n}$$

Solution of Q2 – P.26:

$$\mu = 5 \ \ ; \ \ \sigma = 1 \ \ ; \ \ n = 5$$

(1): $E(\bar{X}) = \mu = 5$

(2): $Var(\bar{X}) = \frac{\sigma^2}{n} = \frac{1}{5} = 0.2$

(3): $n = 16 \rightarrow \frac{\sigma}{\sqrt{n}} = \frac{1}{4}$

$$P(4.5 < \bar{X} < 5.4) = P\left(\frac{4.5-\mu}{\frac{\sigma}{\sqrt{n}}} < Z < \frac{5.4-\mu}{\frac{\sigma}{\sqrt{n}}}\right) = P\left(\frac{4.5-5}{\frac{1}{4}} < Z < \frac{5.4-5}{\frac{1}{4}}\right)$$

$$= P(-2 < Z < 1.6)$$

$$= P(Z < 1.6) - P(Z < -2)$$

$$= 0.9452 - 0.0228 = 0.9224$$

(4): $P(\bar{X} < 5.5) = P\left(Z < \frac{5.5-\mu}{\frac{\sigma}{\sqrt{n}}}\right) = P\left(Z < \frac{5.5-5}{1/4}\right) = P(Z < 2) = 0.9772$

(5): $P(\bar{X} > 4.75) = P\left(Z > \frac{4.75-\mu}{\frac{\sigma}{\sqrt{n}}}\right) = P\left(Z > \frac{4.75-5}{\frac{1}{4}}\right) = P(Z > -1)$

$$= 1 - P(Z < -1) = 1 - 0.1587 = 0.841$$

(6): $P(\bar{X} > a) = 0.1492 \quad ; \quad n = 9$

$$P\left(Z > \frac{a-\mu}{\frac{\sigma}{\sqrt{n}}}\right) = 0.1492$$

$$\Rightarrow 1 - P\left(Z < \frac{a-5}{\frac{1}{3}}\right) = 0.1492$$

$$\Rightarrow P\left(Z < \frac{a-5}{\frac{1}{3}}\right) = 0.8508$$

$$\frac{a-5}{\frac{1}{3}} = 1.04$$

$$a = 5 + \frac{1.04}{3} = 5.347$$

Solution of Q4 – P.26:

$$\mu = 55 \quad ; \quad \sigma = 10 \quad ; \quad n = 64$$

(a) $\bar{X} \sim N\left(\mu, \frac{\sigma}{\sqrt{n}}\right) = \bar{X} \sim N\left(55, \frac{10}{8}\right)$

(b) $E(\bar{X}) = \mu = 55$

(c) $S.D(\bar{X}) = \frac{\sigma}{\sqrt{n}} = \frac{10}{\sqrt{64}} = \frac{10}{8}$

(d) $$P(\bar{X} > 52) = P\left(Z > \frac{52-55}{\frac{10}{8}}\right)$$

$$= P(Z > -2.4)$$

$$= 1 - P(Z < -2.4)$$

$$= 1 - 0.0082 = 0.9918$$

Sampling Distribution: Two Means

$$*\ \bar{X}_1 - \bar{X}_2 \sim N\left(\mu_1 - \mu_2\ ,\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}\right)$$

$$*\ E(\bar{X}_1 - \bar{X}_2) = \mu_1 - \mu_2 \qquad *\ Var(\bar{X}_1 - \bar{X}_2) = \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}$$

Solution of Q1 - P.27:

$$n_1 = 36\,, \mu_1 = 70\,, \sigma_1 = 4$$

$$n_2 = 49\,, \mu_2 = 85\,, \sigma_2 = 5$$

(a): $E(\bar{X}_1) = \mu_1 = 70 \quad \& \quad Var(\bar{X}_1) = \frac{\sigma_1^2}{n_1} = \frac{4^2}{36} = \frac{16}{36}$

(b): $E(\bar{X}_1 - \bar{X}_2) = \mu_1 - \mu_2 = 70 - 85 = -15$

$$Var(\bar{X}_1 - \bar{X}_2) = \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2} = \frac{16}{36} + \frac{25}{49} = 0.955$$

(c): $P(70 < \bar{X}_1 < 71) = P\left(\frac{70-70}{4/6} < Z < \frac{71-70}{4/6}\right) = P(0 < Z < 1.5)$

$$= P(Z < 1.5) - P(Z < 0)$$

$$= 0.9332 - 0.5 = 0.4332$$

(d): $P(\bar{X}_1 - \bar{X}_2 > -16) = P\left(Z > \frac{-16-(-15)}{\sqrt{0.955}}\right) = 1 - P\left(Z < \frac{-16-(-15)}{\sqrt{0.955}}\right)$

$$= 1 - P(Z < -1.02) = 0.8461$$

Solution of Q2 - P.27:

$$n_1 = 25\,, \mu_1 = 100\,, \sigma_1 = 6$$

$$n_2 = 36\,, \mu_2 = 97\,, \sigma_2 = 5$$

$$E(\bar{X}_1 - \bar{X}_2) = \mu_1 - \mu_2 = 100 - 97 = 3$$

$$Var(\bar{X}_1 - \bar{X}_2) = \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2} = \frac{36}{25} + \frac{25}{36} = 2.35$$

(1) $P(\bar{X}_1 > \bar{X}_2 + 6) = P(\bar{X}_1 - \bar{X}_2 > 6)$

$$= P\left(Z > \frac{6-(3)}{\sqrt{2.35}}\right) = P(Z > 2.05)$$

$$= 1 - P(Z < 2.05)$$

$$= 1 - 0.9798 = 0.0202$$

(2) $P(\bar{X}_1 - \bar{X}_2 < 2) = P\left(Z < \frac{2-(3)}{\sqrt{2.35}}\right)$

$$= P(Z < -0.68) = 0.2483$$

Sampling Distribution: Single Proportion

$$* \ \hat{p} \sim N\left(p, \sqrt{\frac{pq}{n}}\right)$$

$$* \ E(\hat{p}) = p \qquad * \ Var(\hat{p}) = \frac{pq}{n}$$

Solution of Q1 – P.28

$$p = 0.2 \ ; \quad n = 5$$

(1): $E(\hat{p}) = p = 0.2$

(2): $Var(\hat{p}) = \frac{pq}{n} = \frac{0.2 \times 0.8}{5} = 0.032$

(3): $\hat{p} \sim N\left(0.2, \sqrt{0.032}\right)$

(4): $P(\hat{p} > 0.25) = P\left(Z > \frac{0.25 - 0.2}{\sqrt{0.032}}\right) = P(Z > 0.28)$

$$= 1 - P(Z < 0.28) = 1 - 0.6103 = 0.3897$$

Sampling Distribution: Two Proportions

$$* \hat{p}_1 - \hat{p}_2 \sim N\left(p_1 - p_2, \sqrt{\frac{p_1 q_1}{n_1} + \frac{p_2 q_2}{n_2}}\right)$$

$$* E(\hat{p}_1 - \hat{p}_2) = p_1 - p_2 \qquad * Var(\hat{p}_1 - \hat{p}_2) = \frac{p_1 q_1}{n_1} + \frac{p_2 q_2}{n_2}$$

Solution of Q1 – P.28:

$$p_1 = 0.25 \;\; ; \;\; n_1 = 5$$

$$p_2 = 0.2 \;\; ; \;\; n_2 = 10$$

(1): $E(\hat{p}_1 - \hat{p}_2) = p_1 - p_2 = 0.25 - 0.2 = 0.05$

(2): $Var(\hat{p}_1 - \hat{p}_2) = \frac{p_1 q_1}{n_1} + \frac{p_2 q_2}{n_2} = \frac{0.25 \times 0.75}{5} + \frac{0.2 \times 0.8}{10} = 0.054$

(3): $\hat{p}_1 - \hat{p}_2 \sim N\left(0.05, \sqrt{0.054}\right)$

(4): $P(0.1 < \hat{p}_1 - \hat{p}_2 < 0.2) = \left(\frac{0.1 - 0.05}{\sqrt{0.054}} < Z < \frac{0.2 - 0.05}{\sqrt{0.054}}\right)$

$$= (0.22 < Z < 0.65)$$

$$P(Z < 0.65) - P(Z < 0.22)$$

$$= 0.7422 - 0.5871 = 0.1551$$

Estimation and Confidence Interval

Estimation and Confidence Interval: Single Mean:

- To find the confidence intervals for a single mean:

  1- $\bar{X} \pm \left(Z_{\frac{\alpha}{2}} \ \frac{\sigma}{\sqrt{n}}\right)$

  2- $\bar{X} \pm \left(t_{\frac{\alpha}{2},n-1} \ \frac{S}{\sqrt{n}}\right)$

- $Z_{\frac{\alpha}{2}} = a \Longrightarrow P(Z > a) = \frac{\alpha}{2}$

- To estimate an error:

$$e = Z_{\frac{\alpha}{2}} \ \frac{\sigma}{\sqrt{n}}$$

- To estimate the sample size with particular error:

$$n = \left(\frac{Z_{\frac{\alpha}{2}} \ \sigma}{e}\right)^2$$

<u>Solution of Q2 – P29:</u>

$$\sigma = 2 \ \ \& \ \ \bar{X} = 4.5 \ \ \& \ \ n = 49$$

(1): $\bar{X} \sim N\left(\mu, \frac{\sigma}{\sqrt{n}}\right) \Longrightarrow \bar{X} \sim N\left(\mu, \frac{2}{7}\right)$

(2): $\hat{\mu} = \bar{X} = 4.5$

(3): $S.E\ (\bar{X}) = \frac{2}{7} = 0.2857$

(4):

$$\bar{X} \pm \left(Z_{\frac{\alpha}{2}} \ \frac{\sigma}{\sqrt{n}}\right)$$

$$4.5 \pm \left(1.96 \times \frac{2}{7}\right)$$

$$95\% \rightarrow \alpha = 0.05$$

$$Z_{\frac{\alpha}{2}} = Z_{0.025} = 1.96$$

The 95% confidence interval is: $(3.94, \ 5.06)$

(5): $The\ upper\ limit = 4.5 + e = 5.2$

and $the\ lower\ limit = 4.5 - e$ = ?

$$4.5 + e = 5.2 \ \Rightarrow \ e = 5.2 - 4.5 \Rightarrow (e = 0.7)$$

Then, $the\ lower\ limit = 4.5 - e = 4.5 - 0.7 = 3.8$

(6): we have the interval:(3.88,5.12):

$$\text{The upper limit} = 5.12$$

$$\Rightarrow \bar{X} + \left(Z_{\frac{\alpha}{2}} \ \frac{\sigma}{\sqrt{n}}\right) = 5.12$$

$$\Rightarrow 4.5 + \left(Z_{\frac{\alpha}{2}} \times \frac{2}{7}\right) = 5.12$$

$$\Rightarrow Z_{\frac{\alpha}{2}} = 2.17 \Rightarrow \frac{\alpha}{2} = 1 - 0.985$$

$$\Rightarrow \frac{\alpha}{2} = 0.015 \Rightarrow \alpha = 0.03$$

Hence, the confidence level is 97%.

<u>Or, we can do the same thing with the lower limit:</u>

$$\text{The lower limit} = 3.88$$

$$\Rightarrow \bar{X} - \left(Z_{\frac{\alpha}{2}} \ \frac{\sigma}{\sqrt{n}}\right) = 3.88$$

$$\Rightarrow 4.5 - \left(Z_{\frac{\alpha}{2}} \times \frac{2}{7}\right) = 3.88$$

$$\Rightarrow Z_{\frac{\alpha}{2}} = 2.17 \Rightarrow \frac{\alpha}{2} = 1 - 0.985$$

$$\Rightarrow \frac{\alpha}{2} = 0.015 \Rightarrow \alpha = 0.03$$

Hence, the confidence level is 97%

(7): $95\% \rightarrow \alpha = 0.05 \quad \rightarrow \quad Z_{\frac{\alpha}{2}} = Z_{0.025} = 1.96$

$$e = \frac{Z_{\frac{\alpha}{2}} \times \sigma}{\sqrt{n}} = \frac{1.96 \times 2}{7} = 0.56$$

(8): $95\% \rightarrow \alpha = 0.05 \quad \rightarrow \quad Z_{\frac{\alpha}{2}} = Z_{0.025} = 1.96 \ \ \& \ \ e = 0.1$

$$n = \left(\frac{Z_{\frac{\alpha}{2}} \times \sigma}{e}\right)^2 = \left(\frac{1.96 \times 2}{0.1}\right)^2 = 1536.64 \simeq 1537$$

Solution of Q3 –P.29:

Data: 3.4 4.8 3.6 3.3 5.6 3.7 4.4 5.2 & 4.8

$$99\% \rightarrow \alpha = 0.01 \rightarrow t_{\frac{\alpha}{2},n-1} = t_{0.005,8} = 3.355.$$

$$\bar{X} \pm \left(t_{\frac{\alpha}{2},n-1} \times \frac{S}{\sqrt{n}}\right)$$

$$= 4.31 \pm \left(3.355 \times \frac{0.84}{3}\right)$$

$$\bar{X} = \frac{\sum x_i}{n} = 4.31$$

$$S^2 = \frac{\sum (x_i - \bar{X})^2}{n-1} = 0.71$$

$$S^2 = 0.71 \Rightarrow S = 0.84$$

The 99% confidence interval is: $(3.37,\ 5.25)$

Solution of Q4 –P.29:

(1)

$$\sigma = 5 \ \ \& \ \ e = 2 \ \ \& \ \ \alpha = 0.10$$

$$\alpha = 0.10 \ \rightarrow \ Z_{\frac{\alpha}{2}} = Z_{0.05} = 1.645$$

$$n = \left(\frac{Z_{\frac{\alpha}{2}} \times \sigma}{e}\right)^2 = \left(\frac{1.645 \times 5}{2}\right)^2 = 16.9 \simeq 17$$

(2) $\sigma = 5 \ \ \& \ \ n = 49 \ \ \& \ \ \bar{X} = 390$

$(i)\ \hat{\mu} = \bar{X} = 390$

$$(ii)\ \ \bar{X} \pm \left(Z_{\frac{\alpha}{2}} \ \frac{\sigma}{\sqrt{n}}\right)$$

$$390 \pm \left(1.96 \ \frac{5}{\sqrt{49}}\right)$$

$$95\% \rightarrow \alpha = 0.05$$

$$Z_{\frac{\alpha}{2}} = Z_{0.025} = 1.96$$

The 95% confidence interval is: $(388.6,\ 391.3)$

Solution of Q5 –P.30:

$$\sigma = 1.4 \ \ \& \ \ e = 0.3 \ \ \& \ \ \alpha = 0.04$$

$$\alpha = 0.04 \ \rightarrow \ Z_{\frac{\alpha}{2}} = Z_{0.02} = 2.055$$

$$n = \left(\frac{Z_{\frac{\alpha}{2}} \times \sigma}{e}\right)^2 = \left(\frac{2.055 \times 1.4}{0.3}\right)^2 = 91.9 \simeq 92$$

Estimation and Confidence Interval: Two Means

To find the confidence intervals for two means:

1- $(\bar{X}_1 - \bar{X}_2) \pm \left(Z_{\frac{\alpha}{2}} \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} \right)$

2- $(\bar{X}_1 - \bar{X}_2) \pm \left(t_{\frac{\alpha}{2}, n_1+n_2-2}\, Sp \sqrt{\frac{1}{n_1} + \frac{1}{n_2}} \right)$

$$Sp^2 = \frac{S_1^2(n_1 - 1) + S_2^2(n_2 - 1)}{n_1 + n_2 - 2}$$

<u>Solution of Question 1(II) – P.31:</u>

$$Theard\ 1: \quad n_1 = 20\,, \bar{X}_1 = 72.8\,, \sigma_1 = 6.8$$

$$Thread\ 2: \quad n_2 = 25\,, \bar{X}_2 = 64.4\,, \sigma_2 = 6.8$$

$$98\% \rightarrow \alpha = 0.02 \rightarrow \quad Z_{\frac{\alpha}{2}} = Z_{0.01} = 2.33$$

$$(\bar{X}_1 - \bar{X}_2) \pm \left(Z_{\frac{\alpha}{2}} \times \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} \right)$$

$$(72.8 - 64.4) \pm \left(2.33 \times \sqrt{\frac{6.8^2}{20} + \frac{6.8^2}{25}} \right)$$

$$8.4 \pm (2.33)(2.04) = (\ 3.65\,, 13.15\ )$$

(1): $The\ lower\ limit\ = 3.65$

(2): $The\ upper\ limit\ = 13.15$

Solution of Q2 – P.31

	First sample	Second sample
Sample size (n)	12	14
Sample mean ($\bar{X}$)	10.5	10
Sample variance (S^2)	4	5

(1): $E(\bar{X}_1 - \bar{X}_2) = \bar{X}_1 - \bar{X}_2 = 10.5 - 10 = 0.5$

(2): $95\% \rightarrow \alpha = 0.05 \rightarrow t_{\frac{\alpha}{2}, n_1+n_2-2} = t_{0.025\,,24} = 2.064,$

$$(\bar{X}_1 - \bar{X}_2) \pm \left(t_{\frac{\alpha}{2}, n_1+n_2-2} \times Sp\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}\right)$$

$$(0.5) \pm \left(2.064 \times 2.13\sqrt{\frac{1}{12} + \frac{1}{14}}\right)$$

$$S_p^2 = \frac{S_1^2(n_1 - 1) + S_2^2(n_2 - 1)}{n_1 + n_2 - 2}$$

$$= \frac{4(11) + 5(13)}{24} = 4.54$$

$$S_p^2 = 4.54 \Rightarrow Sp = 2.13$$

The 95% confidence interval is: $(-1.23,\ 2.23)$

Estimation and Confidence Interval: Single Proportion

* Point estimate for P is: $\frac{x}{n}$

* Interval estimate for P is: $\hat{p} \pm \left(Z_{\frac{\alpha}{2}} \times \sqrt{\frac{\hat{p}\hat{q}}{n}}\right)$

Solution of Q1 - P.33

$$n = 200 \ \ \& \ \ x = 15$$

(1): $\hat{p} = \frac{x}{n} = \frac{15}{200} = 0.075 \rightarrow \ \hat{q} = 0.925$

(2): $95\% \rightarrow \alpha = 0.05 \quad \rightarrow \quad Z_{\frac{\alpha}{2}} = Z_{0.025} = 1.96$

$$\hat{p} \pm \left(Z_{\frac{\alpha}{2}} \times \sqrt{\frac{\hat{p}\hat{q}}{n}}\right)$$

$$= 0.075 \pm \left(1.96 \times \sqrt{\frac{0.075 \times 0.925}{200}}\right)$$

The 95% confidence interval is: $(0.038,\ 0.112)$

Estimation and Confidence Interval: Two Proportions

$$\text{* Point estimate for } P_1 - P_2 = \hat{p}_1 - \hat{p}_2 = \frac{x_1}{n_1} - \frac{x_2}{n_2}$$

$$\text{* Interval estimate for } P_1 - P_2 \text{ is: } (\hat{p}_1 - \hat{p}_2) \pm \left(Z_{\frac{\alpha}{2}} \times \sqrt{\frac{\hat{p}_1\hat{q}_1}{n_1} + \frac{\hat{p}_2\hat{q}_2}{n_2}} \right)$$

Solution of Q 3 – P.34

$$n_1 = 100 \quad x_1 = 15 \quad \rightarrow \hat{p}_1 = \frac{15}{100} = 0.15$$

$$n_2 = 200 \quad x_2 = 20 \rightarrow \hat{p}_2 = \frac{20}{200} = 0.10$$

(1): $\hat{p}_1 - \hat{p}_2 = 0.15 - 0.1 = 0.05$

(2): $95\% \rightarrow \alpha = 0.05 \quad \rightarrow \quad Z_{\frac{\alpha}{2}} = Z_{0.025} = 1.96$

$$(\hat{p}_1 - \hat{p}_2) \pm \left(Z_{\frac{\alpha}{2}} \times \sqrt{\frac{\hat{p}_1\hat{q}_1}{n_1} + \frac{\hat{p}_2\hat{q}_2}{n_2}} \right)$$

$$= (0.05) \pm \left(1.96 \times \sqrt{\frac{(0.15)(0.85)}{100} + \frac{(0.1)(0.9)}{200}} \right)$$

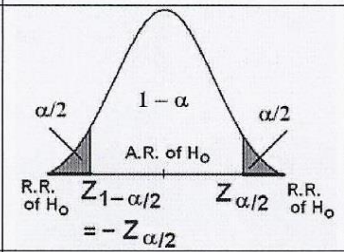
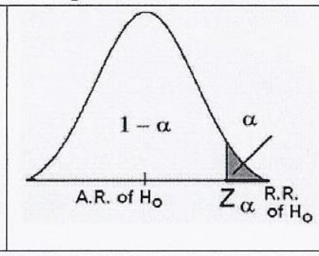
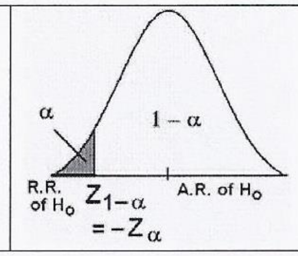
$$= 0.05 \pm \left(1.96 \times \sqrt{0.001725}\right)$$

The 95% confidence interval is: $(-0.031,\ 0.131)$

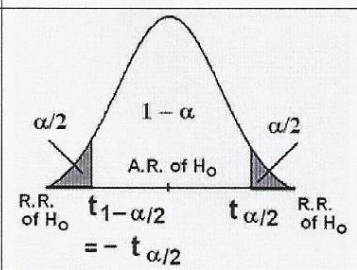
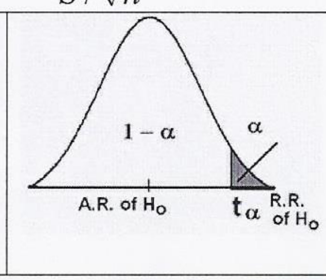
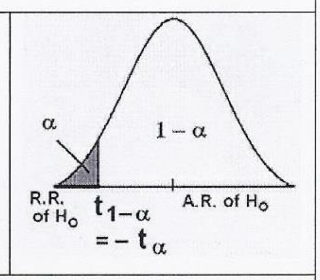
<u>***Hypotheses Testing***</u>

<u>***1-Single Mean***</u>

($if\ \sigma\ known$):

Hypotheses	$H_o: \mu = \mu_o$ $H_1: \mu \neq \mu_o$	$H_o: \mu = \mu_o$ $H_1: \mu > \mu_o$	$H_o: \mu = \mu_o$ $H_1: \mu < \mu_o$
Test Statistic (T.S.)	$Z = \dfrac{\overline{X} - \mu_o}{\sigma / \sqrt{n}} \sim N(0,1)$		
R.R. and A.R. of H_o			
Decision:	Reject H_o (and accept H_1) at the significance level α if:		
	$Z > Z_{\alpha/2}$ or $Z < -Z_{\alpha/2}$ Two-Sided Test	$Z > Z_{\alpha}$ One-Sided Test	$Z < -Z_{\alpha}$ One-Sided Test

($if\ \sigma\ unknown$):

Hypotheses	$H_o: \mu = \mu_o$ $H_1: \mu \neq \mu_o$	$H_o: \mu = \mu_o$ $H_1: \mu > \mu_o$	$H_o: \mu = \mu_o$ $H_1: \mu < \mu_o$
Test Statistic (T.S.)	$T = \dfrac{\overline{X} - \mu_o}{S / \sqrt{n}} \sim t(n-1)$		
R.R. and A.R. of H_o			
Decision:	Reject H_o (and accept H_1) at the significance level α if:		
	$T > t_{\alpha/2}$ or $T < -t_{\alpha/2}$ Two-Sided Test	$T > t_{\alpha}$ One-Sided Test	$T < -t_{\alpha}$ One-Sided Test

Solution of Q1- P.35:

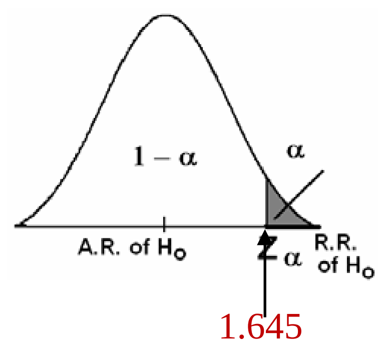
$$\sigma = 2 \quad , \quad n = 49 \quad , \quad \bar{X} = 4.5$$

(1): $H_0: \mu = 5 \quad vs \quad H_1: \mu \neq 5$,

Then the test Statistic (T.S):

$$Z = \frac{\bar{X} - \mu_o}{\sigma/\sqrt{n}} = \frac{4.5 - 5}{2/7} = -1.75$$

(2): $H_0: \mu = 5 \quad vs \quad H_1: \mu > 5$



$$\alpha = 0.05 \rightarrow Z_\alpha = Z_{0.05} = 1.645$$

The Rejection Region (R.R) is $(1.645, \infty)$

(3): $H_0: \mu = 5 \quad vs \quad H_1: \mu > 5, \quad \alpha = 0.05$

$Z = -1.75 \notin R.R = (1.645, \infty)$, Then we accept H_0.

Solution of Q2- P.35:

$$\sigma = 30 \quad , \quad n = 50 \quad , \quad \bar{X} = 750$$

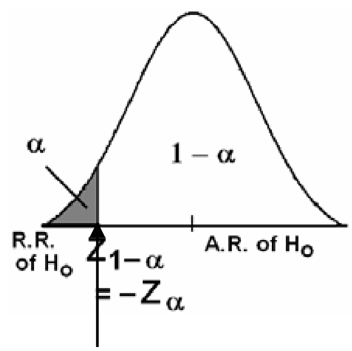
- Hypotheses:

$$H_0: \mu = 740 \quad vs \quad H_1: \mu < 740$$

- Test Statistic (T.S):

$$Z = \frac{\bar{X} - \mu_o}{\sigma/\sqrt{n}} = \frac{750 - 740}{30/\sqrt{50}} = 2.37$$

- Rejection Region (R.R):



-1.645

$$Z_\alpha = Z_{0.05} = 1.645$$

- Decision:

$$Z = 2.37 \notin R.R, then\ we\ accept\ H_0$$

Solution of Q4- P.35:

$$s = 3 \quad , \quad n = 36 \quad , \quad \bar{X} = 15.2 \quad , \quad \alpha = 0.05$$

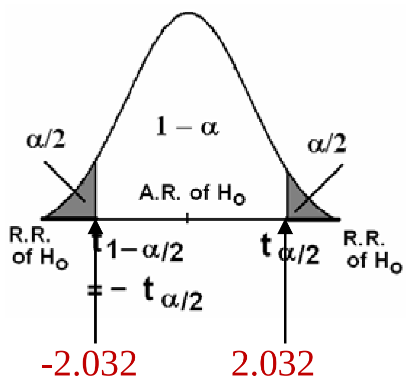
- Hypotheses:

$$H_0: \mu = 15 \quad vs \quad H_1: \mu \neq 15$$

- Test statistic (T.S):

$$t = \frac{\bar{X} - \mu_o}{S/\sqrt{n}} = \frac{15.2 - 15}{3/\sqrt{36}} = 0.4$$

- Rejection Region (R.R):

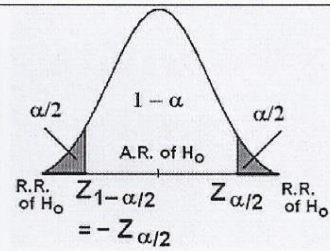
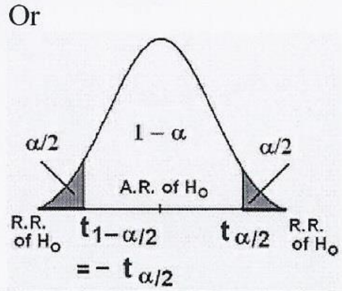
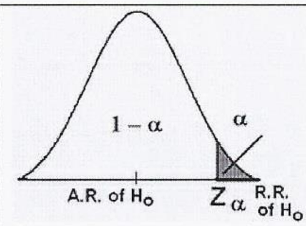
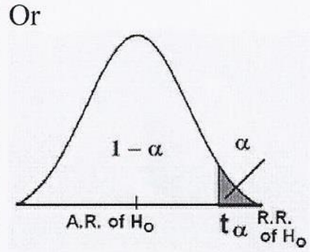
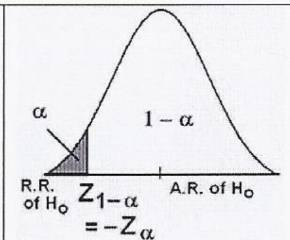
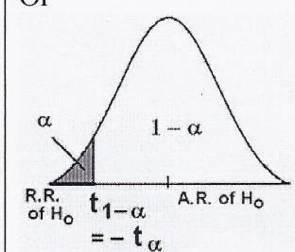


$$t_{\frac{\alpha}{2},n-1} = t_{0.025\,,35} = 2.032$$

- Decision:

$$t = 0.4 \notin R.R, then\ we\ accept\ H_0$$

2-Two Means:

Hypotheses	$H_o: \mu_1 - \mu_2 = d$ $H_1: \mu_1 - \mu_2 \neq d$	$H_o: \mu_1 - \mu_2 = d$ $H_1: \mu_1 - \mu_2 > d$	$H_o: \mu_1 - \mu_2 = d$ $H_1: \mu_1 - \mu_2 < d$
Test Statistic (T.S.)	$Z = \dfrac{(\bar{X}_1 - \bar{X}_2) - d}{\sqrt{\dfrac{\sigma_1^2}{n_1} + \dfrac{\sigma_2^2}{n_2}}} \sim N(0,1)$ {if σ_1^2 and σ_2^2 are known} or $T = \dfrac{(\bar{X}_1 - \bar{X}_2) - d}{S_p\sqrt{\dfrac{1}{n_1} + \dfrac{1}{n_2}}} \sim t(n_1+n_2-2)$ {if $\sigma_1^2 = \sigma_2^2 = \sigma^2$ is unknown}		
R.R. and A.R. of H_o	 Or 	 Or 	 Or 
Decision:	Reject H_o (and accept H_1) at the significance level α if:		
	T.S. $\in$ R.R. Two-Sided Test	T.S. $\in$ R.R. One-Sided Test	T.S. $\in$ R.R. One-Sided Test

$$Sp^2 = \frac{S_1^2(n_1 - 1) + S_2^2(n_2 - 1)}{n_1 + n_2 - 2}$$

Solution of Q1 – P.36:

	First sample	Second sample
Sample size (n)	12	14
Sample mean ($\bar{X}$)	10.5	10
Sample variance (S^2)	4	5

- Hypotheses:

$$H_0: \mu_1 = \mu_2 \qquad or \qquad H_0: \mu_1 - \mu_2 = 0$$

$$H_1: \mu_1 \neq \mu_2 \qquad\qquad H_1: \mu_1 - \mu_2 \neq 0$$

- Test statistic (T.S):

$$t = \frac{(\bar{X}_1 - \bar{X}_2) - d}{Sp\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

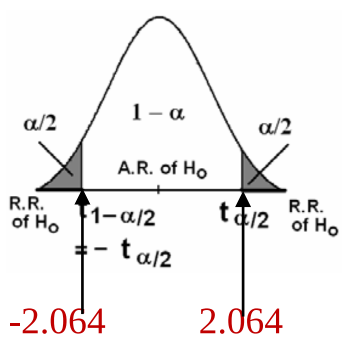
$$= \frac{(10.5 - 10) - 0}{2.13\sqrt{\frac{1}{12} + \frac{1}{14}}} = 0.597$$

$$Sp^2 = \frac{S_1^2(n_1 - 1) + S_2^2(n_2 - 1)}{n_1 + n_2 - 2}$$

$$= \frac{4(11) + 5(13)}{24} = 4.54$$

$$Sp^2 = 4.54 \Rightarrow Sp = 2.13$$

- Rejection Region (R.R):

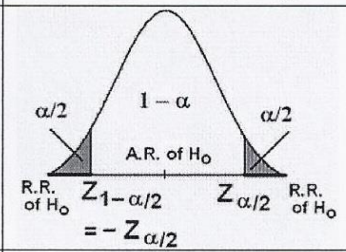
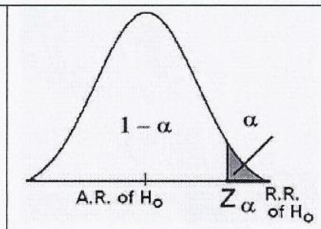
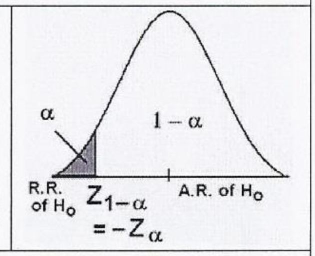


$$\alpha = 0.05 \rightarrow t_{\frac{\alpha}{2}, n_1 + n_2 - 2} = t_{0.025, 24} = 2.064$$

- Decision:

$$t = 0.597 \notin R.R \text{ Then we accept } H_0.$$

3-Single Proportion:

Hypotheses	$H_o: p = p_o$ $H_1: p \neq p_o$	$H_o: p = p_o$ $H_1: p > p_o$	$H_o: p = p_o$ $H_1: p < p_o$
Test Statistic (T.S.)	$Z = \dfrac{\hat{p} - p_o}{\sqrt{\dfrac{p_o q_o}{n}}} = \dfrac{X - np_o}{\sqrt{np_o q_o}} \sim N(0,1) \quad (q_o = 1 - p_o)$		
R.R. and A.R. of H_o			
Decision:	Reject H_o (and accept H_1) at the significance level α if:		
	$Z > Z_{\alpha/2}$ or $Z < -Z_{\alpha/2}$ Two-Sided Test	$Z > Z_{\alpha}$ One-Sided Test	$Z < -Z_{\alpha}$ One-Sided Test

Solution of Q1 – P.37:

$$n = 500 \quad \& \quad x = 150 \ \rightarrow \hat{p} = \frac{150}{500} = 0.3$$

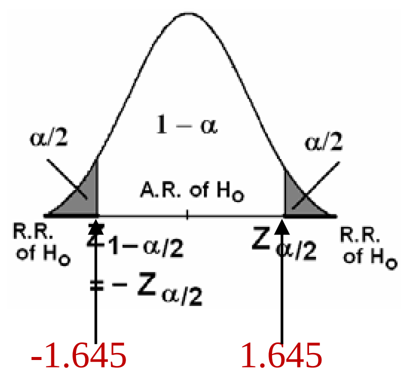
(1): $H_0: p = 0.25 \quad vs \quad H_1: p \neq 0.25$

Then the test Statistic (T.S):

$$Z = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0 q_0}{n}}} = \frac{0.3 - 0.25}{\sqrt{\frac{0.25 \times 0.75}{500}}} = 2.58$$

(2): $H_0: p = 0.25 \quad vs \quad H_1: p \neq 0.25$,

Then the Acceptance Region (A.R) at $\alpha = 0.1$:



$$\alpha = 0.1 \rightarrow Z_{\frac{\alpha}{2}} = Z_{0.05} = 1.645 \rightarrow A.R = (-1.645, 1.645).$$

(3): $H_0: p = 0.25 \quad vs \quad H_1: p \neq 0.25$,

Then the decision at $\alpha = 0.1$:

$Z = 2.58 \notin A.R$, then we reject H_0.

Solution of Q2 – P.37:

$$n = 500 \;\;\&\;\; x = 114 \;\rightarrow \hat{p} = \frac{114}{500} = 0.23$$

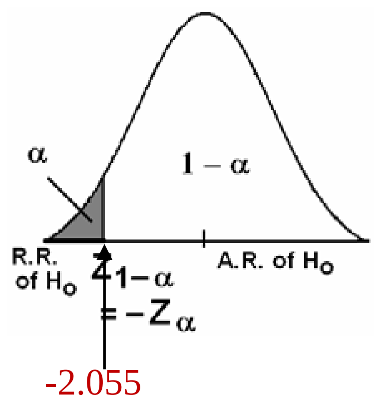
- Hypotheses:

$$H_0: p = 0.2 \quad vs \quad H_1: p < 0.2$$

- Test statistic (T.S):

$$Z = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0 q_0}{n}}} = \frac{0.23 - 0.2}{\sqrt{\frac{0.2 \times 0.8}{500}}} = 1.67$$

- Rejection Region (R.R):

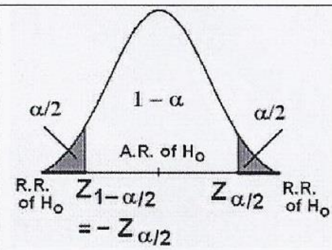
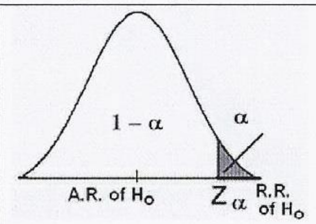
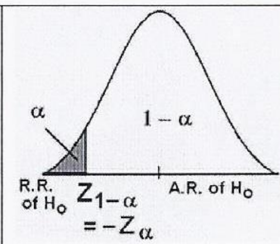


$$98\% \rightarrow \alpha = 0.02 \;\rightarrow Z_\alpha = Z_{0.02} = 2.055$$

- Decision:

$Z = 1.67 \;\notin R.R$, then we accept H_0.

4-Two Proportions:

Hypotheses	$H_o: p_1 - p_2 = 0$ $H_1: p_1 - p_2 \neq 0$	$H_o: p_1 - p_2 = 0$ $H_1: p_1 - p_2 > 0$	$H_o: p_1 - p_2 = 0$ $H_1: p_1 - p_2 < 0$
Test Statistic (T.S.)	$Z = \frac{(\hat{p}_1 - \hat{p}_2)}{\sqrt{\hat{p}\hat{q}\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} \sim N(0,1)$		
R.R. and A.R. of H_o			
Decision:	Reject H_o (and accept H_1) at the significance level α if:		

$$\hat{p} = \frac{x_1 + x_2}{n_1 + n_2}$$

Solution of Q1 – P.38:

$$n_1 = 100 \ \ \& \ \ x_1 = 15 \ \ \rightarrow \ \ \hat{p}_1 = \frac{x_1}{n_1} = \frac{15}{100} = 0.15$$

$$n_2 = 200 \ \ \& \ \ x_2 = 20 \ \ \rightarrow \ \ \hat{p}_2 = \frac{x_2}{n_2} = \frac{20}{200} = 0.10$$

- Hypotheses:

$$H_0: p_1 = p_2 \qquad or \qquad H_0: p_1 - p_2 = 0$$

$$H_1: p_1 > p_2 \qquad\qquad H_1: p_1 - p_2 > 0$$

- Test statistic (T.S):

$$Z = \frac{(\hat{p}_1 - \hat{p}_2)}{\sqrt{\hat{p}\hat{q}\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$$

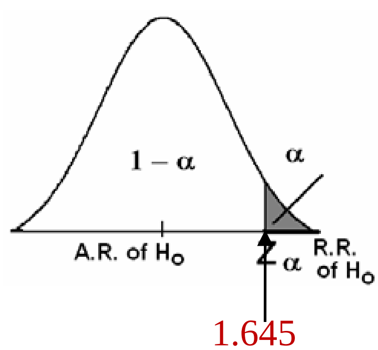
$$= \frac{(0.15 - 0.10)}{\sqrt{(0.12)(0.88)(\frac{1}{100} + \frac{1}{200})}} = 1.26$$

$$\hat{p} = \frac{x_1 + x_2}{n_1 + n_2}$$

$$= \frac{15 + 20}{100 + 200}$$

$$= 0.12$$

- Rejection Region (R.R):



$$\alpha = 0.05 \rightarrow Z_\alpha = Z_{0.05} = 1.645$$

- Decision:

$Z = 1.26 \notin R.R = (\ 1.645, \infty)$, Then we accept H_0.

FOR FINAL

	Single mean	*Two means*	*Single proportion*	*Two proportions*
Sampling Distribution	$\bar{X}\sim N\left(\mu, \frac{\sigma}{\sqrt{n}}\right)$	$\bar{X}_1-\bar{X}_2\sim N\left(\mu_1-\mu_2\ ,\sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}}\right)$	$\hat{p}\sim N\left(p,\sqrt{\frac{pq}{n}}\right)$	$\hat{p}_1-\hat{p}_2\sim N\left(p_1-p_2,\sqrt{\frac{p_1q_1}{n_1}+\frac{p_2q_2}{n_2}}\right)$
Confident Interval	$\bar{X}\pm\left(Z_{\frac{\alpha}{2}}\ \frac{\sigma}{\sqrt{n}}\right)$ σ *known* $\bar{X}\pm\left(t_{\frac{\alpha}{2},n-1}\ \frac{S}{\sqrt{n}}\right)$ σ *unknown*	$(\bar{X}_1-\bar{X}_2)\pm\left(Z_{\frac{\alpha}{2}}\sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}}\right)$ σ_1 *and* σ_2 *known* $(\bar{X}_1-\bar{X}_2)\pm\left(t_{\frac{\alpha}{2},n_1+n_2-2}\ Sp\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}\right)$ σ_1 *and* σ_2 *unknown* $S_P^2=\frac{S_1^2(n_1-1)+S_2^2(n_2-1)}{n_1+n_2-2}$	$\hat{p}\pm\left(Z_{\frac{\alpha}{2}}\times\sqrt{\frac{\hat{p}\hat{q}}{n}}\right)$	$(\hat{p}_1-\hat{p}_2)\pm\left(Z_{\frac{\alpha}{2}}\times\sqrt{\frac{\hat{p}_1\hat{q}_1}{n_1}+\frac{\hat{p}_2\hat{q}_2}{n_2}}\right)$
Testing	$Z=\frac{\bar{X}-\mu_o}{\sigma/\sqrt{n}}$ σ *known* $t=\frac{\bar{X}-\mu_o}{S/\sqrt{n}}$ σ *unknown*	$Z=\frac{(\bar{X}_1-\bar{X}_2)-d}{\sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}}}$ σ_1 & σ_2 *known* $t=\frac{(\bar{X}_1-\bar{X}_2)-d}{Sp\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}}$ σ_1 & σ_2 *unknown* $S_P^2=\frac{S_1^2(n_1-1)+S_2^2(n_2-1)}{n_1+n_2-2}$	$Z=\frac{\hat{p}-p_0}{\sqrt{\frac{p_0q_0}{n}}}$	$Z=\frac{(\hat{p}_1-\hat{p}_2)}{\sqrt{\hat{p}\hat{q}\left(\frac{1}{n_1}+\frac{1}{n_2}\right)}}$ $\hat{p}=\frac{x_1+x_2}{n_1+n_2}$