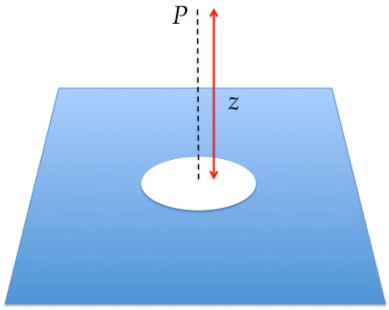
PHYSICS 507 – SPRING 2021
2nd HOMEWORK- Solutions
Dr. V. Lempesis

Hand in: Tuesday 9th of February at 23:59

1. An infinite sheet is charged with a positive surface charged density σ. We open a circular hole of radius R as shown in the Figure. Find the electric field at a distance z above the center of the hole.



Solution:
We can consider the electric field at P as a superposition of two other fields, namely the field from an infinite sheet of a positive surface charge density $+\sigma$ and the field of a negatively charged disk of surface charge density $-\sigma$.

The field of the infinitely charged sheet is given by:

$$\mathbf{E}_{sh} = \frac{\sigma}{2\varepsilon_0}\hat{\mathbf{k}} \quad (1)$$

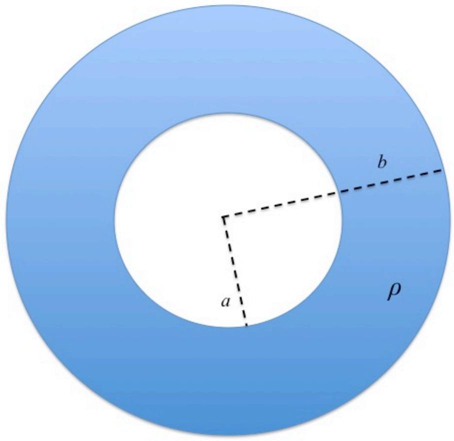
The field created by the negatively charged disk is (see problem 2.7)

$$\mathbf{E}_{sh} = -\frac{\sigma}{2\varepsilon_0}\left(1 - \frac{z}{\sqrt{R^2+z^2}}\right)\hat{\mathbf{k}} \quad (2)$$

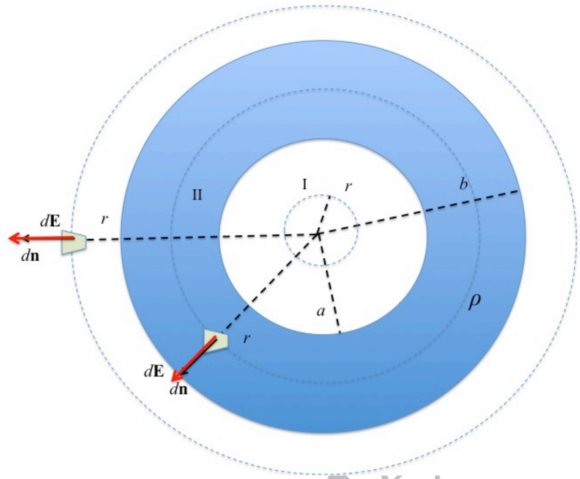
Thus the total field at P is given by:

$$\mathbf{E}_p = \mathbf{E}_{sh} + \mathbf{E}_d = \frac{\sigma}{2\varepsilon_0}\hat{\mathbf{k}} - \frac{\sigma}{2\varepsilon_0}\left(1 - \frac{z}{\sqrt{R^2+z^2}}\right)\hat{\mathbf{k}} = \frac{\sigma}{2\varepsilon_0}\frac{z}{\sqrt{R^2+z^2}}\hat{\mathbf{k}}$$

2. A hollow spherical shell carries a charge density $\rho = k/r^2$ in the region $a \le r \le b$, shown in the figure. (i) Find the electric field in the three regions, (ii) plot the magnitude of the electric field as a function of *r,* anf (iii) find the potential at the center using infinity a your reference point. (iv) Calculate the electrostatic energy of this configuration.



Solution:



(i) We have to apply Gauss' Law for the three regions shown in the figure.

Region I: 0 < *r* < *a*
Inside the Gaussian surface which is shown in the figure there is no charge. Thus

$$\int \mathbf{E}\cdot d\mathbf{A} = \frac{q_{enc}}{\varepsilon_0} \Rightarrow \int \mathbf{E}\cdot d\mathbf{A} = 0 \Rightarrow \mathbf{E} = 0$$

Region II: *a* < *r* < *b*

$$\int \mathbf{E}\cdot d\mathbf{A} = \frac{q_{enc}}{\varepsilon_0} \Rightarrow \int \mathbf{E}\cdot d\mathbf{A} = \frac{q_{enc}}{\varepsilon_0} \Rightarrow \int \mathbf{E}\cdot d\mathbf{A} = \frac{1}{\varepsilon_0}\int_V \rho\, dV \Rightarrow$$

$$\int \mathbf{E}\cdot d\mathbf{A} = \frac{1}{\varepsilon_0}\int_a^r \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \frac{k}{r^2} r^2\, dr \sin\theta\, d\theta\, d\phi$$

Since the problem has a spherical symmetry, the electric field has a radial direction as sown in the figure, thus

$$\int \mathbf{E}\cdot d\mathbf{A} = \frac{1}{\varepsilon_0}\int_a^r \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \frac{k}{r^2} r^2\, dr \sin\theta\, d\theta\, d\phi \Rightarrow \int E\, dA = \frac{4\pi k(r-a)}{\varepsilon_0} \Rightarrow$$

$$E 4\pi r^2 = \frac{4\pi k(r-a)}{\varepsilon_0} \Rightarrow E = \frac{k(r-a)}{\varepsilon_0 r^2}$$

Region III: $r > b$

Applying the Gauss' Law we have

$$\int \mathbf{E}\cdot d\mathbf{A} = \frac{1}{\varepsilon_0}\int_a^b \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \frac{k}{r^2} r^2\, dr \sin\theta\, d\theta\, d\phi \Rightarrow \int E\, dA = \frac{4\pi k(b-a)}{\varepsilon_0} \Rightarrow$$

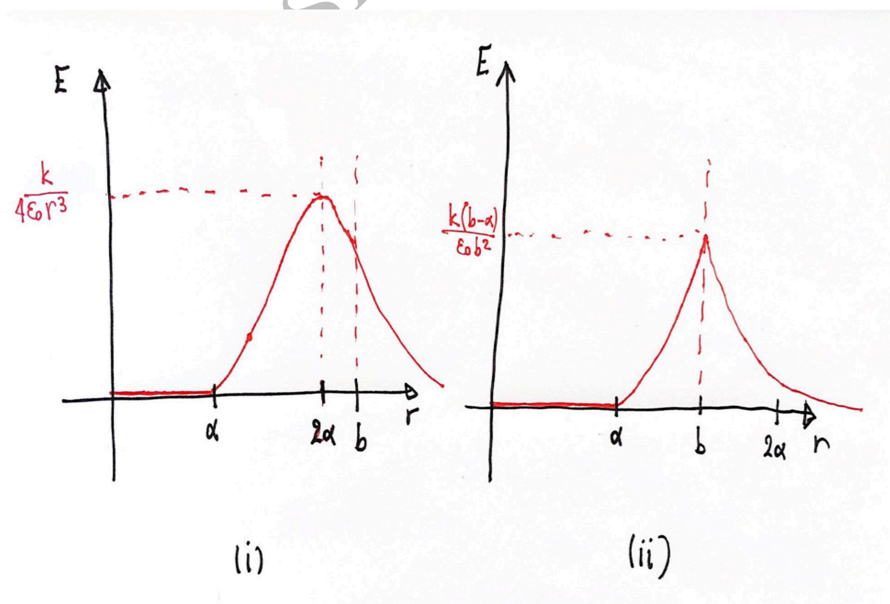
$$E 4\pi r^2 = \frac{4\pi k(b-a)}{\varepsilon_0} \Rightarrow E = \frac{k(b-a)}{\varepsilon_0 r^2}$$

With the direction outwards the center because the charge is positive.

(ii) The plot of the electric field is:

The function $\frac{(r-a)}{r^2}$ has a maximum at $r = 2a$. So it depends now on the relative size of $2a$ with respect to b.

- If $2a < b$ then the plot is (i) (qualitatively):
- If $2a > b$ then the plot is (ii) (qualitatively):



c) For the potential at the center we have:

$$V=-\int_{\infty}^{r}\mathbf{E}\cdot d\mathbf{r}' \Rightarrow V=-\int_{\infty}^{b}E\,dr'-\int_{b}^{a}E\,dr'-\int_{a}^{0}E\,dr' \Rightarrow V=-\frac{k(b-a)}{\varepsilon_0}\int_{\infty}^{b}\frac{1}{r'^2}dr'-\frac{k}{\varepsilon_0}\int_{b}^{a}\frac{(r'-a)}{r'^2}dr' \Rightarrow$$

$$V=-\frac{k(b-a)}{\varepsilon_0}\left[-\frac{1}{r'}\right]_{\infty}^{b}-\frac{k}{\varepsilon_0}\left[\frac{a}{r'}+\ln r'\right]_{b}^{a} \Rightarrow V=\frac{k(b-a)}{\varepsilon_0 b}-\frac{k}{\varepsilon_0}\left[\left(\frac{a}{a}-\frac{a}{b}\right)+\left(\ln(a)-\ln(b)\right)\right] \Rightarrow$$

$$V=\frac{k(b-a)}{\varepsilon_0 b}-\frac{ka}{\varepsilon_0}\left(\frac{1}{a}-\frac{1}{b}\right)-\frac{k}{\varepsilon_0}\ln\left(\frac{a}{b}\right) \Rightarrow V=\frac{k}{\varepsilon_0}\left[\frac{(b-a)}{b}-\frac{(b-a)}{b}-\ln\left(\frac{a}{b}\right)\right] \Rightarrow$$

$$V=\frac{k}{\varepsilon_0}\ln\left(\frac{b}{a}\right)$$

iv) The electrostatic energy of the configuration is given by:

$$W=\frac{\varepsilon_0}{2}\int_{\text{all space}}E^2\,d\tau \Rightarrow W=\frac{\varepsilon_0}{2}\left(\int_{I}0^2\,d\tau+\int_{II}\left(\frac{k(r-a)}{\varepsilon_0 r^2}\right)^2 d\tau+\int_{III}\left(\frac{k(b-a)}{\varepsilon_0 r^2}\right)^2 d\tau\right)$$

$$W=\frac{k^2}{2\varepsilon_0}\left(\int_{II}\frac{(r-a)^2}{r^4}r^2\,dr\sin\theta\,d\theta\,d\phi+\int_{III}\frac{(b-a)^2}{r^4}r^2\,dr\sin\theta\,d\theta\,d\phi\right)$$

$$W=\frac{4\pi k^2}{2\varepsilon_0}\left(\int_{a}^{b}\frac{(r-a)^2}{r^2}dr+(b-a)^2\int_{b}^{\infty}\frac{1}{r^2}dr\right)$$

$$W=\frac{2\pi k^2}{\varepsilon_0}\left\{\left[\left(r-\frac{a^2}{r}-2a\ln r\right)_{a}^{b}\right]-(b-a)^2\left(\frac{1}{r}\right)_{b}^{\infty}\right\}$$

$$W=\frac{2\pi k^2}{\varepsilon_0}\left\{\left[\left(b-\frac{a^2}{b}-2a\ln b\right)-\left(a-\frac{a^2}{a}-2a\ln a\right)\right]-(b-a)^2\left(\frac{1}{\infty}-\frac{1}{b}\right)\right\}$$

$$W=\frac{2\pi k^2}{\varepsilon_0}\left\{\left[(b-a)-a\left(\frac{a-b}{b}\right)-2a\ln\left(\frac{b}{a}\right)\right]+\frac{(b-a)^2}{b}\right\}$$

$$W=\frac{2\pi k^2}{\varepsilon_0}\left\{\left[\frac{b(b-a)-a\left(a-b\right)}{b}-2a\ln\left(\frac{b}{a}\right)\right]+\frac{(b-a)^2}{b}\right\}$$

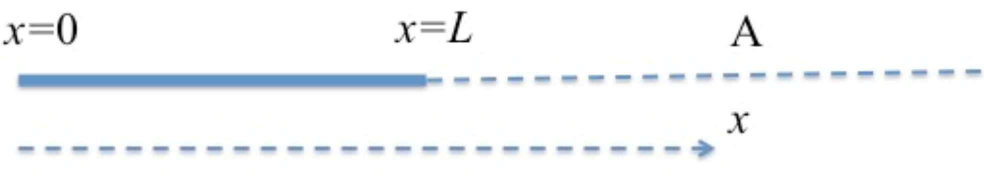
$$W=\frac{2\pi k^2}{\varepsilon_0}\left\{\left[\frac{\left(b+a\right)(b-a)}{b}-2a\ln\left(\frac{b}{a}\right)\right]+\frac{(b-a)^2}{b}\right\}$$

$$W=\frac{2\pi k^2}{\varepsilon_0}\left\{\left[\frac{b^2-a^2}{b}-2a\ln\left(\frac{b}{a}\right)\right]+\frac{(b-a)^2}{b}\right\}$$

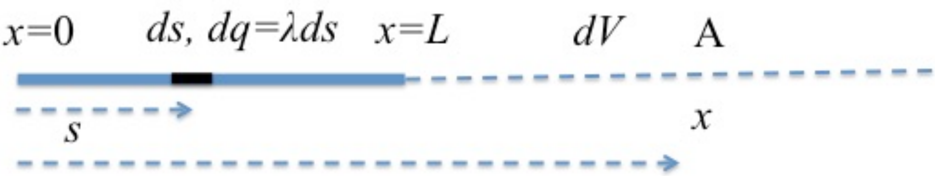
$$W = \frac{2\pi k^2}{\varepsilon_0}\left[\frac{b^2 - a^2 + b^2 + a^2 - 2ab}{b} - 2a\ln\left(\frac{b}{a}\right)\right]$$

$$W = \frac{4\pi k^2}{\varepsilon_0}\left[b - a - a\ln\left(\frac{b}{a}\right)\right]$$

3. An infinitely long wire carries positive charge with uniform linear charge density λ. Find the electric **potential** at point A at a distance x from the origin using the formula $V = \frac{1}{4\pi\varepsilon_0}\int_L \frac{\lambda(\mathbf{r}')}{r} dl'$. Other solution is not going to be accepted.



Solution:



The elementary charge dq creates an elementary potential dV at point A given by:

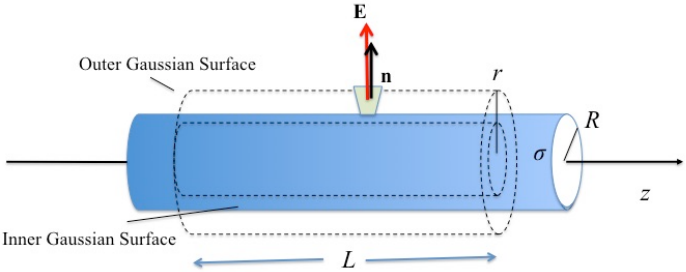
$$dV = \frac{dq}{4\pi\varepsilon_0 (x - s)} = \frac{\lambda ds}{4\pi\varepsilon_0 (x - s)}$$

$$V = \int_{s=0}^{L} dV = \int_{s=0}^{L} \frac{\lambda ds}{4\pi\varepsilon_0 (x - s)} = \frac{\lambda}{4\pi\varepsilon_0}\int_{s=0}^{L} \frac{ds}{(x - s)}$$

$$= -\frac{\lambda}{4\pi\varepsilon_0}\left[\ln(x - s)\right]_{s=0}^{L} = -\frac{\lambda}{4\pi\varepsilon_0}\left(\ln(x - L) - \ln(x)\right) = -\frac{\lambda}{4\pi\varepsilon_0}\ln\left(\frac{x - L}{x}\right) = \frac{\lambda}{4\pi\varepsilon_0}\ln\left(\frac{x}{x - L}\right)$$

4. Use Gauss' Law to find out the electric field inside and outside a very long hollow cylinder which carries a positive charge with surface charge density σ. Check that the results are consistent with $E^{\perp}_{\text{above}} - E^{\perp}_{\text{below}} = \dfrac{1}{\varepsilon_0}\sigma$.

Solution:



Due to the infinite length of the cylinder the electric field will have a radial outward (positive charge) direction. There are two regions:

A) Inside the cylinder. If we draw a cylindrical Gaussian (Inner) surface it does not contain charge since the cylindrical shell is hollow.
B) Outside the cylinder we draw an cylindrical Gaussuan surface of radius r. In this case we have:

$$\int \mathbf{E}\cdot d\mathbf{A} = \frac{q_{enc}}{\varepsilon_0} \Rightarrow \int \mathbf{E}\cdot \mathbf{n}\, dA = \frac{\sigma 2\pi RL}{\varepsilon_0} \Rightarrow E\int dA = \frac{\sigma 2\pi RL}{\varepsilon_0} \Rightarrow$$

$$E 2\pi r L = \frac{\sigma 2\pi RL}{\varepsilon_0} \Rightarrow E = \frac{\sigma R}{\varepsilon_0 r}$$

Thus the electric field vector is $\mathbf{E} = \dfrac{\sigma R}{\varepsilon_0 r}\hat{\rho}$

For the boundary condition

$$\mathbf{E}^{\perp}_{\text{above}} - \mathbf{E}^{\perp}_{\text{below}} = \left.\frac{\sigma R}{\varepsilon_0 r}\right|_{r=R} \hat{\rho} - 0 \Rightarrow \mathbf{E}^{\perp}_{\text{above}} - \mathbf{E}^{\perp}_{\text{below}} = \frac{\sigma}{\varepsilon_0}\hat{\rho} \Rightarrow E^{\perp}_{\text{above}} - E^{\perp}_{\text{below}} = \frac{\sigma}{\varepsilon_0}$$