BAYES RULE

Bayes Rule

Q1.Upon arrival at a hospital's emergency room, patients are categorized according to their condition as critical, serious, or stable. In the past year:

(i) 10% of the emergency room patients were critical;
(ii) 30% of the emergency room patients were serious;
(iii) the rest of the emergency room patients were stable;
(iv) 40% of the critical patients died;
(vi) 10% of the serious patients died; and
(vii) 1% of the stable patients died.

Given that a patient survived, calculate the probability that the patient was categorized as serious upon arrival.

(A) 0.06 (B) 0.29 (C) 0.30 (D) 0.39 (E) 0.64

Solution :

Let C=critical ; SE= serious ; ST= stable ; D= died ; SU= survive

We are given that P(C)=0.1 , P(SE)=0.3 , P(ST)=1-(0.1+0.3)=0.6 ,

P(D|C)=0.4 , P(D|SE)=0.1 , P(D|ST)=0.01

Therefore,

$$P(SE|SU) = \frac{P(SU|SE)\,P(\mathrm{SE})}{P(SU|C)\,P(C) + P(SU|SE)\,P(\mathrm{SE}) + P(SU|ST)\,P(\mathrm{ST})}$$

$$= \frac{(0.9)(0.3)}{(0.6)(0.1) + (0.9)(0.3) + (0.99)(0.6)} = 0.29$$

Q2. An actuary studied the likelihood that different types of drivers would be involved in at least one collision during any one-year period. The results of the study are:

Type of driver	Percentage of all drivers	Probability of at least one collision
Teen	8%	0.15
Young adult	16%	0.08
Midlife	45%	0.04
Senior	31%	0.05
Total	100%	

Given that a driver has been involved in at least one collision in the past year, calculate the probability that the driver is a young adult driver.

(A) 0.06 (B) 0.16 (C) 0.19 (D) 0.22 (E) 0.25

Solution : H.W

Let

C = Event of a collision

T = Event of a teen driver

Y = Event of a young adult driver

M = Event of a midlife driver

S = Event of a senior driver

Then,

$$P(Y|C) = \frac{P(C|Y)\,P(\mathrm{Y})}{P(C|T)\,P(T)\ +\ P(C|Y)\,P(\mathrm{Y}) + P(C|M)\,P(\mathrm{M}) + P(C|S)\,P(\mathrm{S})}$$

$$= \frac{(0.08)(0.16)}{(0.15)(0.08)\ (0.08)(0.16)\ (0.04)(0.45)\ (0.05)(0.31)} = 0.22$$

Q3. A blood test indicates the presence of a particular disease 95% of the time when the disease is actually present. The same test indicates the presence of the disease 0.5% of the time when the disease is not actually present. One percent of the population actually has the disease. Calculate the probability that a person actually has the disease given that the test indicates the presence of the disease.

(A) 0.324 (B) 0.657 (C) 0.945 (D) 0.950 (E) 0.995

Solution :

Let Y = positive test result

D = disease is present

Then,

$$P(D|Y) = \frac{P(Y|D)\,P(\mathrm{D})}{P(Y|D)\,P(D)\ +\ P(Y|D^C)\,P(\mathrm{D}^{\mathrm{C}})} = \frac{(0.95)(0.01)}{(0.95)(0.01)\ (0.005)(0.99)} = 0.657$$

Q4. The probability that a randomly chosen male has a blood circulation problem is 0.25. Males who have a blood circulation problem are twice as likely to be smokers as those who do not have a blood circulation problem.

Calculate the probability that a male has a blood circulation problem, given that he is a smoker.

(A) 1/4 (B) 1/3 (C) 2/5 (D) 1/2 (E) 2/3

Solution :

Let:

S = Event of a smoker

C = Event of a circulation problem

Then we are given that $P[C]\ =\ 0.25$ and $P[S\mid C]\ =\ 2\,P[S\mid C^C]$

Then,

$$P(C|S) = \frac{P(S|C)P(C)}{P(S|C)P(C) + P(S|C^C)P(C^C)} = \frac{2\,P(S|C^C)P(C)}{2\,P(S|C^C)P(C) + P(S|C^C)P(C^C)}$$

$$= \frac{2P(C)}{2P(C) + P(C^C)} = \frac{2\,(0.25)}{2(0.25) + 0.75} = \frac{2}{5}$$

RANDOM VARIABLES, DISTRIBUTIONS AND EXPECTATIONS

DISCRETE DISTRIBUTIONS:

Q1. Consider the experiment of flipping a balanced coin three times independently.

Let X= Number of heads – Number of tails.

a) List the elements of the sample space S.

$$S = \{HHH, HHT, HTH, THH, TTH, THT, HTT, TTT\}, n = 2^3 = 8$$

b) Assign a value x of X to each sample point.

S	TTT	TTH	THT	THH	HHH	HTT	HTH	HHT
X	0-3= **-3**	1-2= **-1**	1-2= **-1**	2-1= **1**	3-0= **3**	1-2= **-1**	2-1=**1**	2-1=**1**

c) Find the probability distribution function of X.

x	-3	-1	1	3	total
f(x)	1/8	3/8	3/8	1/8	1

d) Find $P(\, X \leq 1\,) = F(1) = \left(\frac{1}{8}\right) + \left(\frac{3}{8}\right) + \left(\frac{3}{8}\right) = \frac{7}{8}$

e) Find $P(\, X \;<\; 1\,) = \left(\frac{1}{8}\right) + \left(\frac{3}{8}\right) = \frac{4}{8} = \frac{1}{2}$

f) Find $\mu = E(X)$

$$E(X) = \sum_x x.f(x) \;=\; -3\left(\frac{1}{8}\right) - 1\left(\frac{3}{8}\right) + 1\left(\frac{3}{8}\right) + 3\left(\frac{1}{8}\right) \;=\; 0$$

g) Find $s^2 = Var(X)$

$$Var(X) = E(X^2) - \left(E(X)\right)^2 = \sum_x x^2.f(x) - \left(E(X)\right)^2$$

$$= 9\left(\frac{1}{8}\right) + 1\left(\frac{3}{8}\right) + 1\left(\frac{3}{8}\right) + 9\left(\frac{1}{8}\right) - 0 = 3$$

Q2. Q3.: H.W

Q4. Let X be a random variable with the following probability distribution:

x	-3	6	9
f(x)	0.1	0.5	0.4

a) Find the mean (expected value) of X , $\mu = E(X)$

$$E(X) = \sum_x x.f(x) = -3(0.1) + 6(0.5) + 9(0.4) = 6.3$$

b) Find $E(X^2)$

$$E(X^2) = \sum_x x^2.f(x) = (-3)^2\,(0.1) + 6^2(0.5) + 9^2(0.4) = 51.3$$

c) Find the variance of X, $Var\,(X) = \sigma_x^2$

$$\sigma_x^2 = E(X^2) - \left(E(X)\right)^2 = 51.3 - (6.3)^2 = 11.61$$

d) Find the mean of 2X+1, $E(2X+1) = \mu_{2X+1}$

$$= 2E(X) + E(1) = 2(6.3) + 1 = 13.6$$

e) Find the variance of 2X+1, $Var(2X+1) = \sigma^2_{2X+1}$

$$= 2^2\,Var(X) + Var(1) = 4(11.61) + 0 = 46.44$$

Q5. Which of the following is a probability distribution function:

(a) $f(x) = \frac{x+1}{10}$; $x = 0,1,2,3,4$

$f(0) = \left(\frac{1}{10}\right) = 0.1 < 1$; $f(1) = \left(\frac{2}{10}\right) = 0.2 < 1$; $f(2) = \left(\frac{3}{10}\right) = 0.3 < 1$;
$f(3) = \left(\frac{4}{10}\right) = 0.4 < 1$; $f(4) = \left(\frac{5}{10}\right) = 0.5 < 1$

$$\sum f(x) = \frac{1+2+3+4+5}{10} = 1.5 \neq 1 \quad \therefore f(x)\ is\ not\ PDF$$

(b) $f(x) = \frac{x-1}{5}$; $x = 0,1,2,3,4$

$f(0) = \frac{-1}{5} < 0 \quad \therefore f(x)$ *is not PDF.*

(c) $f(x) = \frac{1}{5}$; $x = 0,1,2,3,4$

$f(0) = f(1) = f(2) = f(3) = f(4) = \frac{1}{5}$

$\sum f(x) = \frac{1+1+1+1+1}{5} = 1 \qquad \therefore f(x)$ *is PDF*

(d) $f(x) = \frac{5-x^2}{6}$; $x = 0,1,2,3$

$f(0) = \frac{5}{6} < 1$; $f(1) = \frac{4}{6} < 1$; $f(2) = \frac{1}{6} < 1$; $f(3) = -\frac{4}{6} < 0$

since $f(3) < 0$, $f(x)$ *is not PDF*

Q6. Let X be a discrete random variable with the probability distribution function:

$$f(x) = kx \quad for\ x = 1, 2, and\ 3.$$

(i) Find the value of k. $we\ know\ that\ \sum_x f(x) = 1$

$$\sum_x kx = 1 \rightarrow k + 2k + 3K = 1 \rightarrow 6k = 1 \rightarrow k = 1/6$$

$$\boldsymbol{f(x) = \frac{x}{6}\ ; x = 1, 2, 3}$$

(ii) Find the cumulative distribution function (CDF), F(x)

$F(1) = P(X \leq 1) = P(X = 1) = f(1) = 1/6$

$F(2) = P(X \leq 2) = P(X = 1) + P(X = 2) = 3/6$

$F(3) = P(X \leq 3) = P(X = 1) + P(X = 2) + P(X = 3) = 1$

$$F(x) = P(X \leq x) = \begin{cases} 0 & x < 1 \\ 1/6 & 1 \leq x < 2 \\ 3/6 & 2 \leq x < 3 \\ 1 & x \geq 3 \end{cases}$$

(iii) Using the CDF, F(x), find P $(0.5 < X \leq 2.5)$.

$P\ (0.5 < X \leq 2.5) = P(X \leq 2.5) - P(X \leq 0.5)$

$= F(2.5) - F(0.5) = \left(\frac{3}{6}\right) - 0 = \frac{3}{6} = \frac{1}{2}$

$Or\ by\ use\ \ f(x)$: $P\ (0.5 < X \leq 2.5) = f(1) + f(2) = \frac{1}{6} + \frac{2}{6} = \frac{1}{2}$

Q7. Let X be a random variable with cumulative distribution function (CDF) given by:

$$F(x) = \begin{cases} 0 & x < 0 \\ 0.25 & 0 \leq x < 1 \\ 0.6 & 1 \leq x < 2 \\ 1 & x \geq 2 \end{cases}$$

(a) Find the probability distribution function of X, $f(x)$.

$$f(x) = F(x) - F(x-1)$$

$f(0) = 0.25 - 0 = 0.25$

$f(1) = 0.6 - 0.25 = 0.35$

$f(2) = 1 - 0.6 = 0.4$

$$f(x) = \begin{cases} 0.25 & x = 0 \\ 0.35 & x = 1 \\ 0.4 & x = 2 \\ 0 & otherwise \end{cases}$$

(b) Find $P(1 \leq X < 2)$. (using both $f(x)$ and $F(x)$)

By using $f(x)$:

$P(1 \leq X < 2) = P(X = 1) = f(1) = 0.35$

By using *F(X)* :

$P(1 \leq X < 2) = F(2-1) - F(1-1) = F(1) - F(0) = 0.6 - 0.25 = 0.35$

(c) Find $P(X > 2)$. (using both f(x) and F(x))

By using $f(x)$:

$P(X > 2) = 1 - P(X \leq 2) = 1 - [f(0) + f(1) + f(2)] = 0$

By using *F(X)* : $P(X > 2) = 1 - F(2) = 1 - 1 = 0$

Find $P(1 < X \leq 2) = F(2) - F(1) = 1 - 0.6 = 0.4$

Find $P(1 \leq X \leq 2) = F(2) - F(1-1) = 1 - 0.25 = 0.75$

Find $P(1 < X < 2) = F(2-1) - F(1) = F(1) - F(1) = 0$

Note:

$\boldsymbol{P(a \leq X < b) = F(b-1) - F(a-1)}$

$\boldsymbol{P(a < X \leq b) = F(b) - F(a)}$

$\boldsymbol{P(a \leq X \leq b) = F(b) - F(a-1)}$

$\boldsymbol{P(a < X < b) = F(b-1) - F(a)}$

Q8. Consider the random variable X with the following probability distribution function:

X	0	1	2	3
f(x)	0.4	*c*	0.3	0.1

The value of ***C*** is

(A) 0.125 (B) 0.2 (C) 0.1 (D) 0.125 (E) - 0.2

$we\ know\ that\ \sum_x f(x) = 1$

$0.4 + C + 0.3 + 0.1 = 1 \rightarrow 0.8 + C = 1 \rightarrow C = 1 - 0.8 = 0.2$

Q9. The probability distribution for company **A** is given by:

x	1	2	3
f(x)	0.3	0.4	0.3

and for company **B** is given by:

y	0	1	2	3	4
f(y)	0.2	0.1	0.3	0.3	0.1

Show that the variance of the probability distribution for company B is greater than that of company A.

Company A:

$E(X) = \sum_x x.f(x) = 1(0.3) + 2(0.4) + 3(0.3) = 2$

$E(X^2) = \sum_x x^2.f(x) = 1^2(0.3) + 2^2(0.4) + 3^2(0.3) = 4.6$

$Var(X) = E(X^2) - (E(X))^2 = 4.6 - 2^2 = \mathbf{0.6}$

Company B:

$E(Y) = 0(0.2) + 1(0.1) + 2(0.3) + 3(0.3) + 4(0.1) = 2$

$E(Y^2) = 0^2(0.2) + 1^2(0.1) + 2^2(0.3) + 3^2(0.3) + 4^2(0.1) = 5.6$

$Var(Y) = E(Y^2) - (E(Y))^2 = 5.6 - 2^2 = \mathbf{1.6}$

$\therefore\ var(Y) > var(X)$