King Saud University: Mathematics Department M-254
Semester II (2nd Midterm Exam-Solution) 1439-1440 H
Maximum Marks=25 Time Allowed: 90 Mins.

Questions : (5 + 5 + 5 + 5 + 5)

Q1: Use LU-factorization with Doolittle's method ($l_{ii} = 1$) to find the value of α for which the following linear system has infinitely many solutions, and write down this solution.

$$\begin{array}{rcrcrcr} x_1 & + & x_2 & & & = & 5/9 \\ 3x_1 & + & \alpha x_2 & + & 5x_3 & = & 0 \\ & & 7x_2 & + & 3x_3 & = & -1 \end{array}$$

Solution. Using Simple Gauss-elimination method, we can easily find fatorization of A as

$$A = LU = \begin{pmatrix} 1 & 1 & 0 \\ 3 & \alpha & 5 \\ 0 & 7 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 7/(\alpha-3) & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & (\alpha-3) & 5 \\ 0 & 0 & (3\alpha-44)/(\alpha-3) \end{pmatrix}.$$

Since

$$|A| = |U| = (\alpha-3)(3\alpha-44)/(\alpha-3) = 3\alpha - 44, \quad \alpha \neq 3.$$

So $|A| = 0$, gives, $\alpha = 44/3$ and for this value of α we have infinitely many solutions. By solving the lower-triangular system $L\mathbf{y} = \mathbf{b}$ of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 3/5 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 5/9 \\ 0 \\ -1 \end{pmatrix},$$

we obtained the solution $\mathbf{y} = [5/9, -5/3, 0]^T$. Now solving the upper-triangular system $U\mathbf{x} = \mathbf{y}$ of the form

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 35/3 & 5 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5/9 \\ -5/3 \\ 0 \end{pmatrix}.$$

If we choose $x_3 = t \in R,\ t \neq 0$, then, $x_2 = (-1/7 - 3t/7)$ and $x_1 = (44/9 + 3t/7)$, the required solutions. •

Q2: Rearrange the following linear system of equations

$$\begin{array}{rcrcrcr} x_1 & + & 6x_2 & - & 3x_3 & = & 4 \\ 2x_1 & + & 2x_2 & + & 6x_3 & = & 7 \\ 5x_1 & + & 2x_2 & - & x_3 & = & 6 \end{array}$$

such that the convergence of Jacobi iterative method is guaranteed. Then, use the initial solution $\mathbf{x^{(0)}} = [0,0,0]^T$, compute the second approximation $\mathbf{x^{(2)}}$. Also, compute an error bound for the error $\|\mathbf{x} - \mathbf{x^{(10)}}\|$.

Solution. For the guarantee convergence of iterative methods, the system must be SDD form, so rearrange the given system in the following form

$$\begin{array}{rcrcrcr} 5x_1 & + & 2x_2 & - & x_3 & = & 6 \\ x_1 & + & 6x_2 & - & 3x_3 & = & 4 \\ 2x_1 & + & 2x_2 & + & 6x_3 & = & 7 \end{array}$$

The Jacobi iterative formula for the given system is

$$\begin{array}{rcrcrcl} x_1^{(k+1)} & = & 0.2(6 & - & 2x_2^{(k)} & + & x_3^{(k)}) \\ x_2^{(k+1)} & = & 0.1667(4 & - & x_1^{(k)} & + & 3x_3^{(k)}) \\ x_3^{(k+1)} & = & 0.1667(7 & - & 2x_1^{(k)} & - & 2x_2^{(k)}) \end{array}$$

Starting with $\mathbf{x^{(0)}} = [0,0,0]^T$, we obtain the first and the second approximations as

$$\mathbf{x^{(1)}} = [\mathbf{1.200}, \mathbf{0.667}, \mathbf{1.167}]^{\mathbf{T}} \quad \text{and} \quad \mathbf{x^{(2)}} = [\mathbf{1.167}, \mathbf{1.050}, \mathbf{0.544}]^{\mathbf{T}}.$$

Since we know that error bound formula for $k = 10$ is

$$\|\mathbf{x} - \mathbf{x^{(10)}}\| \le \frac{\|T_J\|^{10}}{1 - \|T_J\|} \|\mathbf{x}^{(1)} - \mathbf{x^{(0)}}\|.$$

$$T_J = -D^{-1}(L+U) = -\begin{pmatrix} 5 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{pmatrix}^{-1} \begin{pmatrix} 0 & 2 & -1 \\ 1 & 0 & -3 \\ 2 & 2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -2/5 & 1/5 \\ -1/6 & 0 & 3/6 \\ -2/6 & -2/6 & 0 \end{pmatrix}.$$

$$\|T_J\|_\infty = \max\{3/5, 4/6, 4/6\} = 4/6 = 2/3 < 1.$$

$$\|\mathbf{x} - \mathbf{x^{(10)}}\| \le \frac{(2/3)^{10}}{1 - 2/3} \left\| \begin{pmatrix} 1.200 \\ 0.667 \\ 1.167 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\| \le \frac{(2/3)^{10}}{1 - 2/3}(1.2) = 0.0628.$$

Q3: If $\mathbf{x}^* = [-1.99, 2.99]^T$ is an approximate solution of the linear system $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1/2 \end{bmatrix} \qquad \text{and} \qquad \mathbf{b} = \begin{bmatrix} 1 \\ -0.5 \end{bmatrix},$$

then find an upper bound for the relative error.

Solution. We can easily find the inverse of the given matrix as

$$A^{-1} = \begin{bmatrix} -1 & 2 \\ 2 & -2 \end{bmatrix}.$$

Then the l_∞-norm of both matrices A and A^{-1} are

$$\|A\|_\infty = 2 \quad \text{and} \quad \|A^{-1}\|_\infty = 4,$$

and so the condition number of the matrix can be computed as follows:

$$K(A) = \|A\|_\infty \||A^{-1}\|_\infty = (2)(4) = 8.$$

The residual vector (by taking $n = 2$) can be calculated as

$$\mathbf{r} = \mathbf{b} - A_2\mathbf{x}^* = \begin{pmatrix} 1 \\ -0.5 \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ 1 & 0.5 \end{pmatrix} \begin{pmatrix} -1.99 \\ 2.99 \end{pmatrix} = \begin{pmatrix} 0.000 \\ -0.005 \end{pmatrix},$$

and it gives $\|\mathbf{r}\|_\infty = 0.005$. Now using formula, we obtain

$$\frac{\|\mathbf{x} - \mathbf{x}^*\|}{\|\mathbf{x}\|} \leq K(A)\frac{\|\mathbf{r}\|}{\|\mathbf{b}\|} = (8)\frac{0.005}{1} = 0.0400,$$

which is the required upper bound for the relative error. •

Q4: Let $f(x) = \sqrt{x - x^2}$ and $p_2(x)$ be the quadratic Lagrange interpolating polynomial which interpolates f at $x_0 = 0, x_1 = \alpha$ and $x_2 = 1$. Find the largest value of α, in the interval $(0, 1)$, for which

$$f(0.5) - p_2(0.5) = -0.25.$$

Solution. Consider the quadratic Lagrange interpolating polynomial as follows:

$$f(x) = p_2(x) = L_0(x)f(x_0) + L_1(x)f(x_1) + L_2(x)f(x_2).$$

At the given values of $x_0 = 0, x_1 = \alpha, x_2 = 1$, we have, $f(0) = 0, f(\alpha) = \sqrt{\alpha - \alpha^2}$ and $f(1) = 0$, gives

$$p_2(x) = L_0(x)(0) + L_1(x)(\sqrt{\alpha - \alpha^2}) + L_2(x)(0),$$

where

$$L_1(x) = \frac{(x-0)(x-1)}{(\alpha-0)(\alpha-1)} = \frac{x^2 - x}{\alpha^2 - \alpha}.$$

Thus

$$p_2(x) = \frac{x^2 - x}{\alpha^2 - \alpha}\sqrt{\alpha - \alpha^2} \quad \text{and} \quad p_2(0.5) = \frac{-0.25}{\alpha^2 - \alpha}\sqrt{\alpha - \alpha^2} = \frac{0.25}{\sqrt{\alpha - \alpha^2}}.$$

Given

$$f(0.5) - p_2(0.5) = -0.25, \quad \text{gives} \quad p_2(0.5) = f(0.5) + 0.25 = 0.5 + 0.25 = 0.75,$$

so

$$\frac{0.25}{\sqrt{\alpha - \alpha^2}} = 0.75, \quad \text{or} \quad \sqrt{\alpha - \alpha^2} = \frac{1}{3}.$$

Thus, taking square on both sides, we get

$$\alpha - \alpha^2 = \frac{1}{9}, \quad \text{or} \quad 9\alpha^2 - 9\alpha + 1 = 0.$$

Solving this equation for α, we get, $\alpha = 0.127322$ or $\alpha = 0.872678$. Thus $\alpha = 0.872678$, the required largest value of α in the given interval $(0, 1)$. •

Q5: Let $f(x) = (x+1)\ln(x+1)$ be the function defined over the interval $[1, 2]$. Compute the error bound for fifth degree Lagrange interpolating polynomial for equally spaced data points for the approximation of $(2.9 \ln 2.9)$.

Solution. For the fifth degree Lagrange polynomial, we have $h = \dfrac{2-1}{5} = 0.2$, so using the following points:

$$x_0 = 1, x_1 = 1.2, x_2 = 1.4, x_3 = 1.6, x_4 = 1.8, x_5 = 2.$$

and $x = 1.9$.
For error bound of fifth degree Lagrange polynomial, we use the formula

$$|E_5| \leq \frac{M}{6!}|(x-x_0)(x-x_1)(x-x_2)(x-x_3)(x-x_4)(x-x_5)|,$$

where

$$M = \max_{1 \leq x \leq 2} |f^{(6)}(x)|.$$

The six derivatives of the given function $f(x) = (x+1)\ln(x+1)$ are as follows:

$$f'(x) = 1 + \ln(x+1), \qquad f''(x) = \frac{1}{x+1}, \qquad f'''(x) = -\frac{1}{(x+1)^2},$$

$$f^{(4)}(x) = \frac{2}{(x+1)^3}, \qquad f^{(5)}(x) = \frac{-6}{(x+1)^4}, \qquad f^{(6)}(x) = \frac{24}{(x+1)^5}.$$

Thus

$$M = \max_{1 \leq x \leq 2} |f^{(6)}(x)| = \max_{1 \leq x \leq 2} \left| \frac{24}{(x+1)^5} \right| = \frac{24}{32} = \frac{3}{4}.$$

So

$$|E_5| \leq \frac{(3/4)(9.4500e-004)}{720} = 9.8437e-007.$$