King Saud University: Mathematics Department Math-254
Second MidTerm Examination Semester II 1438-1439 H
Time: 120 min. Max. Marks 25

Question 1: (6)

Use the simple Gaussian elimination method to find all values of α and β for which the following linear system is consistent or inconsistent. Also, find the solution of the consistent system.

$$\begin{array}{rcrcrcr} 2x_1 & - & x_2 & + & 3x_3 & = & 1 \\ 4x_1 & + & 2x_2 & + & 2x_3 & = & 2\alpha \\ 2x_1 & + & x_2 & + & x_3 & = & \beta \end{array}$$

Question 2: (7)

Consider the following matrix and its inverse as follows:

$$A = \begin{pmatrix} 10 & -1 & 0 \\ -1 & 10 & -2 \\ 0 & -2 & 10 \end{pmatrix} \quad \text{and} \quad A^{-1} = \begin{pmatrix} 48/475 & 1/95 & 1/475 \\ 1/95 & 2/19 & 2/95 \\ 1/475 & 2/95 & 99/950 \end{pmatrix}.$$

Show that Gauss-Seidel method converges faster than Jacobi method for the linear system $A\mathbf{x} = [9, 7, 6]^T$. If an approximate solution for this system is $x^* = [0.97, 0.91, 0.74]^T$, then find the relative error.

Question 3: (6)

Find α for which the following matrix A is singular using LU decomposition by Doolittle's method ($l_{ii} = 1$).

$$A = \begin{bmatrix} 1 & 2 & \alpha \\ 2 & 7 & 3\alpha \\ \alpha & 3\alpha & 4 \end{bmatrix}.$$

Then use the smallest positive integer value of α to find the solution of the linear system $A\mathbf{x} = [1, 0, 3]^T$ using Doolittle's method.

Question 4: (6)

Let $f(x) = \dfrac{1}{x}$ be defined in the interval $[2, 4]$ and $x_0 = 2,\ x_1 = 2.5,\ x_2 = 4$. Compute the value of the unknown point $\eta \in (2, 4)$ in the error formula of the quadratic Lagrange interpolating polynomial for the approximation of $f(3)$ using the given points. Also, compute an error bound for the corresponding error.

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Question 1: (6)

Use the simple Gaussian elimination method to find all values of α and β for which the following linear system is consistent or inconsistent. Also, find the solution of the consistent system.

$$\begin{array}{rcrcrcl} 2x_1 & - & x_2 & + & 3x_3 & = & 1 \\ 4x_1 & + & 2x_2 & + & 2x_3 & = & 2\alpha \\ 2x_1 & + & x_2 & + & x_3 & = & \beta \end{array}$$

Solution. Writing the given system in the augmented matrix form

$$\left(\begin{array}{rrrr} 2 & -1 & 3 & 1 \\ 4 & 2 & 2 & 2\alpha \\ 2 & 1 & 1 & \beta \end{array}\right).$$

$$\left(\begin{array}{rrrr} 2 & -1 & 3 & 1 \\ 0 & 4 & -4 & 2\alpha-2 \\ 0 & 2 & -2 & \beta-1 \end{array}\right).$$

$$\left(\begin{array}{rrrr} 2 & -1 & 3 & 1 \\ 0 & 4 & -4 & 2\alpha-2 \\ 0 & 0 & 0 & \beta-\alpha \end{array}\right).$$

We finished with the second column. So third row of the equivalent upper-triangular system is

$$0x_1 + 0x_2 + 0x_3 = \beta - \alpha. \tag{1}$$

The linear system is inconsistent if

$$\beta - \alpha \neq 0.$$

The linear system is consistent if

$$\beta - \alpha = 0.$$

Thus

$$\begin{array}{rcrcrcl} 2x_1 & - & x_2 & + & 3x_3 & = & 1 \\ & & 4x_2 & - & 4x_3 & = & 2\alpha - 2 \end{array}$$

in three unknowns. Let $x_3 = t$,

$$x_2^* = \alpha/2 - 1/2 + t; \quad x_1^* = \frac{1}{2}(1 + \alpha/2 - 1/2 - 2t).$$

Hence

$$\mathbf{x}^* = [\frac{1}{2}(1/2 + \alpha/2 - 2t3), 1/2\alpha - 1/2 + t, t]^T,$$

is an approximation solution of consistent system for any value of t. •

Question 2:

Consider the following matrix and its inverse as follows:

$$A = \begin{pmatrix} 10 & -1 & 0 \\ -1 & 10 & -2 \\ 0 & -2 & 10 \end{pmatrix} \quad \text{and} \quad A^{-1} = \begin{pmatrix} 48/475 & 1/95 & 1/475 \\ 1/95 & 2/19 & 2/95 \\ 1/475 & 2/95 & 99/950 \end{pmatrix}.$$

Show that Gauss-Seidel method converges faster than Jacobi method for the linear system $A\mathbf{x} = [9, 7, 6]^T$. If an approximate solution for this system is $x^* = [0.97, 0.91, 0.74]^T$, then find the relative error.

Solution.

$$T_J = -D^{-1}(L+U) = -\begin{pmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{pmatrix}^{-1} \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & -2 \\ 0 & -2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{10} & 0 \\ \frac{1}{10} & 0 & \frac{2}{10} \\ 0 & \frac{2}{10} & 0 \end{pmatrix}.$$

$$\|T_J\|_\infty = \max\left\{\frac{1}{10}, \frac{3}{10}, \frac{2}{10}\right\} = \frac{3}{10} = 0.3 < 1.$$

$$T_G = -(D+L)^{-1}U = -\begin{pmatrix} 10 & 0 & 0 \\ -1 & 10 & 0 \\ 0 & -2 & 10 \end{pmatrix}^{-1} \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix},$$

$$T_G = -\begin{pmatrix} \frac{1}{10} & 0 & 0 \\ \frac{1}{100} & \frac{1}{10} & 0 \\ \frac{1}{500} & \frac{1}{50} & \frac{1}{10} \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{10} & 0 \\ 0 & \frac{1}{100} & \frac{1}{5} \\ 0 & \frac{1}{500} & \frac{1}{25} \end{pmatrix}.$$

$$\|T_G\|_\infty = \max\left\{\frac{1}{10}, \frac{21}{100}, \frac{26}{500}\right\} = \frac{21}{100} = 0.21 < 1.$$

Since $\|T_G\|_\infty < \|T_J\|_\infty$, which shows that Gauss-Seidel method will converge faster than Jacobi method for the given linear system.

The l_∞-norm of both given matrices are

$$\|A\|_\infty = 13 \quad \text{and} \quad \|A^{-1}\|_\infty = 13/95, \quad K(A) = \|A\|_\infty \|A^{-1}\|_\infty = (13)(13/95) = 1.7789.$$

$$\mathbf{r} = \mathbf{b} - A\mathbf{x}^* = \begin{pmatrix} 9 \\ 7 \\ 6 \end{pmatrix} - \begin{pmatrix} 10 & 0 & 0 \\ -1 & 10 & 0 \\ 0 & -2 & 10 \end{pmatrix} \begin{pmatrix} 0.97 \\ 0.91 \\ 0.74 \end{pmatrix} = \begin{pmatrix} 0.21 \\ 0.35 \\ 0.42 \end{pmatrix},$$

$$\|\mathbf{r}\|_\infty = 0.42, \quad \|\mathbf{b}\|_\infty = 9.$$

$$\frac{\|\mathbf{x} - \mathbf{x}^*\|}{\|\mathbf{x}\|} \leq (1.7789)\frac{(0.42)}{9} = 0.0830.$$

Question 3:

Find α for which the following matrix A is singular using LU decomposition by Doolittle's method ($l_{ii} = 1$).

$$A = \begin{bmatrix} 1 & 2 & \alpha \\ 2 & 7 & 3\alpha \\ \alpha & 3\alpha & 4 \end{bmatrix}.$$

Then use the smallest positive integer value of α to find the solution of the linear system $A\mathbf{x} = [1, 0, 3]^T$ using Doolittle's method.

Solution. We use Simple Gauss-elimination method to convert the following matrix of the given system by using the multiples $m_{21} = 2, m_{31} = \alpha$ and $m_{32} = \alpha/3$,

$$A = \begin{pmatrix} 1 & 2 & \alpha \\ 2 & 3 & \alpha \\ 0 & 0 & (12 - 4\alpha^2)/3 \end{pmatrix} \equiv U.$$

Thus LU-factorization of A is

$$A = \begin{pmatrix} 1 & 2 & \alpha \\ 2 & 7 & 3\alpha \\ \alpha & 3\alpha & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ \alpha & \alpha/3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & \alpha \\ 2 & 3 & \alpha \\ 0 & 0 & (12 - 4\alpha^2)/3 \end{pmatrix} = LU.$$

$$|A| = |U| = 1 \times 3 \times (12 - 4\alpha^2)/3 = 0, \quad \alpha = \pm\sqrt{3}.$$

Now using $\alpha = 1$ and solving the first system $L\mathbf{y} = \mathbf{b}$ for unknown vector $\mathbf{y}$, that is

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 1/3 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}.$$

Performing forward substitution yields

$$y_1 = 1, \quad y_2 = -2, \quad y_3 = 8/3.$$

Then solving the second system $U\mathbf{x} = \mathbf{y}$ for unknown vector $\mathbf{x}$, that is

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 0 & 0 & 8/3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 8/3 \end{pmatrix}.$$

Performing backward substitution yields

$$x_1 = 2, \quad x_2 = -1, \quad x_3 = 1.$$

Question 4:

Let $f(x) = \dfrac{1}{x}$ be defined in the interval $[2,4]$ and $x_0 = 2,\ x_1 = 2.5,\ x_2 = 4$. Compute the value of the unknown point $\eta \in (2,4)$ in the error formula of the quadratic Lagrange interpolating polynomial for the approximation of $f(3)$ using the given points. Also, compute an error bound for the corresponding error.

Solution. Consider the quadratic Lagrange interpolating polynomial as follows:

$$p_2(x) = L_0(x)f(x_0) + L_1(x)f(x_1) + L_2(x)f(x_2),$$

At the given values of $x_0 = 2, x_1 = 2.5, x_2 = 4$, we have, $f(2) = 1/2, f(2.5) = 1/2.5$, and $f(4) = 1/5$, so using $x = 3$, we have

$$f(3) \approx p_2(3) = (1/2)L_0(3) + (1/2.5)L_1(3) + (1/5)L_2(3).$$

Then

$$\begin{aligned} L_0(3) &= \frac{(3-2.5)(3-4)}{(2-2.5)(2-4)} = -\frac{1}{2}, \\ L_1(3) &= \frac{(3-2)(3-4)}{(2.5-2)(2.5-4)} = \frac{4}{3}, \\ L_2(3) &= \frac{(3-2)(3-2.5)}{(4-2)(4-2.5)} = \frac{1}{6}. \end{aligned}$$

So

$$f(3) \approx p_2(3) = (1/2)(-1/2) + (1/2.5)(4/3) + (1/5)(1/6) = 0.3166,$$

which is the required approximation of $f(3)$ by the quadratic interpolating polynomial. The error is

$$f(3) - p_2(3) = 0.333 - 0.3166 = 0.0167.$$

Since the error formula of the quadratic Lagrange polynomial is

$$E = f(x) - p_2(x) = \frac{f'''(\eta)}{3!}(x-x_0)(x-x_1)(x-x_2), \quad \eta \in I,$$

and the third derivative of f is

$$f'(\eta) = -1/\eta^2, \quad f''(\eta) = 2/\eta^3, \quad f'''(\eta) = -6/\eta^4.$$

Thus

$$0.0167 = f(3) - p_2(3) = \left(\frac{(3-2)(3-2.5)(3-4)}{6}\right)\frac{(-6)}{(\eta^4)} = \frac{(0.5)}{(\eta^4)},$$

and solving for η, we get

$$\eta^4 = 29.9401, \quad \eta^2 = 5.4718, \quad \eta = 2.3392 \in (2,4),$$

required value of the unknown point η. •