King Saud University: Mathematics Department M-254
Semester II (1st Midterm Exam) 1438-1439 H
Max Marks=25 Time Allowed: 90 Mins.

Questions : (6 + 6 + 7 + 6)

Q1: Show that the iterative procedure for evaluating $N^{\frac{1}{p}}$ by using secant method can be written

$$x_{n+1} = \frac{x_n x_{n-1}(x_n^{p-1} - x_{n-1}^{p-1}) + N(x_n - x_{n-1})}{x_n^p - x_{n-1}^p}.$$

Then use it to find the second approximation x_3 of the square root of 9 using the initial approximations $x_0 = 2,\ x_1 = 2.5$. Compute the absolute error.

Q2: Consider the nonlinear equation $e^x - 1 = x$, which has a multiple root.
Use a quadratic convergent method to find the second approximation x_2 of this root using the initial approximation $x_0 = 0.5$.

Q3: Consider solving the nonlinear equation $x^3 - 1 = 3x$ in the interval $[1, 2]$.

(a) Show that the iterative scheme $x_{n+1} = (3x_n + 1)^{\frac{1}{3}},\ n \geq 0$ is suitable for solving this equation.

(b) Use this iterative scheme to compute the third approximation x_3 when $x_0 = 1.0$.

(c) Compute an error bound of your approximation.

Q4: Consider the iterative scheme $x_{n+1} = x_n + \lambda(1 - 2e^{-x_n})$.

(a) Show that this iterative scheme converges to the root $\alpha = \ln(2)$ for $\lambda = -1$.

(b) Find the values of λ for which the scheme converges.

(c) Find the values of λ giving a quadratic convergence of the scheme in (b).

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Questions : (6 + 6 + 7 + 6)

Q1: Show that the iterative procedure for evaluating $N^{\frac{1}{p}}$ by using secant method can be written

$$x_{n+1} = \frac{x_n x_{n-1}(x_n^{p-1} - x_{n-1}^{p-1}) + N(x_n - x_{n-1})}{x_n^p - x_{n-1}^p}.$$

Then use it to find the second approximation x_3 of the square root of 9 using the initial approximations $x_0 = 2,\ x_1 = 2.5$. Compute the absolute error.

Solution. We shall compute $x = N^{1/p}$ by finding a positive root for the nonlinear equation

$$x^p - N = 0,$$

where $N > 0$ is the number whose root is to be found. If $f(x) = 0$, then $x = \alpha = N^{1/p}$ is the exact zero of the function

$$f(x) = x^p - N.$$

Since the secant formula is

$$x_{n+1} = x_n - \frac{(x_n - x_{n-1})f(x_n)}{f(x_n) - f(x_{n-1})}, \qquad n \geq 1.$$

Hence, assuming the initial estimates to the root, say, $x = x_0, x = x_1$, and by using the Secant iterative formula, we have

$$x_2 = x_1 - \frac{(x_1 - x_0)(x_1^p - N)}{(x_1^p - N) - (x_0^p - N)} = x_1 - \frac{(x_1 - x_0)(x_1^p - N)}{(x_1 - x_0)(x_1 + x_0)} = \frac{x_1 x_0(x_1^{p-1} - x_0^{p-1}) + N(x_1 - x}{x_1^p - x_0^p}$$

In general, we have

$$x_{n+1} = \frac{x_n x_{n-1}(x_n^{p-1} - x_{n-1}^{p-1}) + N(x_n - x_{n-1})}{x_n^p - x_{n-1}^p}, \qquad n = 1, 2, \ldots,$$

the Secant formula for approximation of the square root of number N. Now using this formula for approximation of the square root of $N = 9$, taking $x_0 = 2$ and $x_1 = 2.5$, we have

$$x_2 = 3.1111 \qquad \text{and} \qquad x_3 = 2.9901.$$

Hence

$$\text{Absolute Error} = |9^{1/2} - x_3| = |3 - 2.9901| = 0.0099,$$

is the possible absolute error.

Q2: Consider the nonlinear equation $e^x - 1 = x$, which has a multiple root. Use a quadratic convergent method to find the second approximation x_2 of this root using the initial approximation $x_0 = 0.5$.

Solution. Since $f(x) = e^x - x - 1$, we have $f'(x) = e^x - 1$ and $f''(x) = e^x$. Now using the Modified Newton's formula (second modified)

$$x_{n+1} = x_n - \frac{f(x_n)f'(x_n)}{[f'(x_n)]^2 - [f(x_n)][f''(x_n)]}, \qquad n \geq 0,$$

we have

$$x_{n+1} = x_n - \frac{(e^{x_n} - x_n - 1)(e^{x_n} - 1)}{[e^{x_n} - 1]^2 - (e^{x_n} - x_n - 1)(e^{x_n})}, \qquad n \geq 0.$$

For $n = 0$ and the initial approximation $x_0 = 0.5$, we have

$$x_1 = x_0 - \frac{(e^{x_0} - x_0 - 1)(e^{x_0} - 1)}{[e^{x_0} - 1]^2 - (e^{x_0} - x_0 - 1)(e^{x_0})} = -0.0493,$$

and

$$x_2 = x_1 - \frac{(e^{x_1} - x_1 - 1)(e^{x_1} - 1)}{[e^{x_1} - 1]^2 - (e^{x_1} - x_1 - 1)(e^{x_1})} = -0.0004,$$

which is the required second approximation.

Q3: Consider solving the nonlinear equation $x^3 - 1 = 3x$ in the interval $[1, 2]$.

(a) Show that the iterative scheme $x_{n+1} = (3x_n + 1)^{\frac{1}{3}}$, $n \geq 0$ is suitable for solving this equation.

(b) Use this iterative scheme to compute the third approximation x_3 when $x_0 = 1.0$.

(c) Compute an error bound of your approximation.

Solution. Since, we observe that $f(1)f(2) < 0$, then the solution we seek is in the interval $[1, 2]$.

(a) For $g(x) = (3x + 1)^{\frac{1}{3}}$, g is increasing function of x, as $g(1) = (4)^{\frac{1}{3}} = 1.5874$ and $g(2) = (7)^{\frac{1}{3}} = 1.9129$ both lie in the interval $[1, 2]$. Thus $g(x) \in [1, 2]$, for all $x \in [1, 2]$. Also, we have $g'(x) = 1/(3x + 1)^{\frac{2}{3}} < 1$, for all x in the given interval $[1, 2]$., so from fixed-point theorem the g(x) has a unique fixed-point.

(b) using the given initial approximation $x_0 = 1.5$, we have the other approximations as

$$x_1 = g(x_0) = 1.5874, \quad x_2 = g(1.5874) = 1.7928, \quad x_3 = g(1.7928) = 1.8545.$$

(c) Since $a = 1$ and $b = 2$, then the value of k can be found as follows

$$k_1 = |g'(1)| = |1/(4)^{\frac{2}{3}}| = 0.3969 \quad \text{and} \quad k_2 = |g'(2)| = |1/(7)^{\frac{2}{3}}| = 0.2733,$$

which give $k = \max\{k_1, k_2\} = 0.3969$. Thus using the error formula (**??**), we have

$$|\alpha - x_3| \leq \frac{(0.3969)^3}{1 - 0.3969}|1.5874 - 1.0| = 0.0609.$$

Q4: Consider the iterative scheme $x_{n+1} = x_n + \lambda(1 - 2e^{-x_n})$.

(a) Show that this iterative scheme converges to the root $\alpha = \ln(2)$ for $\lambda = -1$.

(b) Find the values of λ for which the scheme converges.

(c) Find the values of λ giving a quadratic convergence of the scheme in (b).

Solution. (a) Given $\lambda = -1$,

$$g(x) = x - (1 - 2e^{-x}),$$

and at fixed-point $\alpha = \ln(2)$, we have

$$g(\ln(2)) = \ln(2).$$

Also

$$g'(x) = 1 - 2e^{-x}, \quad |g'(\ln(2))| = |1 - 2e^{-\ln(2)}| = |1 - 1| = 0,$$

so the iterative scheme converges to the root $\alpha = \ln(2)$.

(b) Given

$$g(x) = x + \lambda(1 - 2e^{-x}), \quad g'(x) = 1 + \lambda(2e^{-x}).$$

Since

$$|g'(\ln(2))| < 1, \quad |1 + \lambda(2e^{-\ln(2)})| < 1.$$

so

$$|1 + \lambda| < 1, \quad \text{gives} \quad -2 < \lambda < 0.$$

(c) Given

$$g'(\ln(2)) = 0, \quad \text{gives} \quad 1 + \lambda = 0, \quad \lambda = -1.$$

Note that

$$g''(x) = -2\lambda e^{-x} \quad \text{and} \quad g''(\ln(2)) = -\lambda \neq 0.$$