King Saud University: Mathematics Department Math-254
Second Semester 1432-33 H Final Examination
Maximum Marks = 50 Time: 180 mins.

Name of the Student:——————————— **I.D. No.** ———————

Name of the Teacher:——————————— **Section No.** ———————

Note: Check please the total number of pages are Eight (8).
(20 Multiple choice questions and Four (4) Full questions)

The Answer Tables for Q.1 to Q.20 : Marks: 1.5 for each one ($1.5 \times 20 = 30$)

Ps. : Mark {a, b, c or d} for the correct answer in the box.

Q. No.	1	2	3	4	5	6	7	8	9	10
a,b,c,d										

Q. No.	11	12	13	14	15	16	17	18	19	20
a,b,c,d										

Quest. No.	Marks
Q. 1 to Q. 20	
Q. 21	
Q. 22	
Q. 23	
Q. 24	
Total	

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a,b,c,d	a	b	d	a	c	c	b	d	a	b	a	c	d	d	b	b	c	c	d	a

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Q. No.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
a,b,c,d	b	c	a	c	d	a	d	c	b	a	d	b	c	a	d	d	a	b	c	d

Ps. : Mark {a, b, c or d} for the correct answer in the box. **(MATh)**

Q. No.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
a,b,c,d	d	a	c	b	a	b	c	a	d	c	b	a	b	c	a	a	d	d	a	b

Ps. : Mark {a, b, c or d} for the correct answer in the box. **(MATH)**

Q. No.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
a,b,c,d	c	b	b	d	b	d	a	b	c	d	c	d	a	b	c	c	b	a	b	c

Question 1: Let $f(\alpha) = 0$ and x_n be an approximation for a computed by the bisection method, then the error $|\alpha - x_n|$ is bounded by:

(a) $\frac{b-a}{2^n}$ (b) $\frac{b+a}{2^n}$ (c) $\frac{a-b}{2^n}$ (d) $\frac{b-a}{3^n}$

Question 2: Let $x = g(x)$ the sufficient condition for the uniqueness of the fixed point is:

(a) $|g'(x)| < 1$ (b) $|g(x)| < 1$ (c) $g'(x) < 1$ (d) $|g'(x)| > 1$

Question 3: When using Newton's method with $x_0 = 1$ for solving the equation $\cos(\pi x - \pi) - x = 0$, the first approximation x_1 is:

(a) 1 (b) 1.55 (c) 1.5 (d) 1.35

Question 4: When using secant method with $x_0 = 1$ and $x_1 = 2$ for solving the equation $\cos(\pi x - \pi) - x = 0$, the approximation x_2 is:

(a) 1 (b) 1.55 (c) -1.5 (d) -1

Question 5: The l_∞-norm of Jacobian matrix of the nonlinear system $x^2 - y^2 = 1,\ xy = 1$ at the point $(1, -1)$ is:

(a) 2 (b) 1 (c) 4 (d) 3

Question 6: The rate of convergence the iterative scheme $x_{n+1} = \frac{1}{2}(x_n^2 + 1) - \ln x_n,\ n \geq 0$ to $\alpha = 1$ is:

(a) Order 2 (b) Order 3 (c) Order 4 (d) Order 1

Question 7: Let $\alpha = 0$ is a root for the equation $\ln(x+1) = x$. This root is a:

(a) multiple root with $m = 2$ (b) simple root (c) multiple root with $m = 3$
(d) multiple root with $m = 4$

Questions 8 - 10 are concerned with Linear System $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & 0.5 \\ -2 & 1 \end{bmatrix}, \quad A^{-1} = \begin{bmatrix} 0.5 & -0.25 \\ 1 & 0.5 \end{bmatrix}, \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}.$$

Question 8: In the LU factorization with $u_{ii} = 1,\ i = 1, 2$ of the matrix A, the matrix L is given by:

(a) $\begin{bmatrix} 1 & 0 \\ -2 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 \\ 1 & -2 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 0 \\ 0.5 & -1 \end{bmatrix}$

Question 9: The residual vector $\mathbf{r}$ for this system with respect to the approximate solution $\hat{x} = [2, 0]^T$ is:

(a) $\begin{bmatrix} 2 \\ 6 \end{bmatrix}$ (b) $\begin{bmatrix} 5 \\ 2 \end{bmatrix}$ (c) $\begin{bmatrix} 2.5 \\ 5.5 \end{bmatrix}$ (d) $\begin{bmatrix} 6 \\ 2.5 \end{bmatrix}$

Question 10: The error $\|\mathbf{x} - \hat{\mathbf{x}}\|_\infty$, where $\hat{x} = [2, 0]^T$, is bounded by:

(a) 9 (b) 8.25 (c) 7.5 (d) 10

Question 11: Using Newton divided difference formula we construct the polynomial $p(x) = \alpha + 0.693x - 0.148x(x-1)$ which interpolate the function $f(x) = \ln(x+1)$ at the number $0, 1$, and 2. Then the value of α is:

(a) 0 (b) 0.5 (c) 1 (d) 1.5

Question 12: Consider using the polynomial given in Question 11 to approximate $\ln(2.5)$. The error related to this approximation is bounded by:

(a) $\frac{1}{8}$ (b) $\frac{1}{64}$ (c) $\frac{1}{32}$ (d) $\frac{1}{16}$

Question 13: When using the two-point forward formula with $h = 0.2$ for approximating the value of $f'(1)$, where $f(x) = \ln(x+1)$, we have the computed approximation (accurate to 4 decimal places):

(a) 0.4766 (b) 0.4966 (c) 0.4666 (d) 0.4866

Question 14: The error related to the approximation given in Question 13 is bounded by:

(a) $\frac{1}{40}$ (b) $\frac{1}{25}$ (c) $\frac{1}{30}$ (d) $\frac{1}{35}$

Question 15: When using simple trapezoidal rule for approximating the integral $\int_1^{1.5} \frac{1}{x}\,dx$, we have the computed approximation:

(a) $\frac{5}{12}$ (b) $\frac{7}{12}$ (c) $\frac{5}{14}$ (d) $\frac{9}{14}$

Question 16: If $f(0) = 3, f(1) = \frac{\alpha}{2}, f(2) = \alpha$, and Simpson's rule for $\int_0^2 f(x)\,dx$ gives 2, then the value of α is:

(a) 1.0 (b) 2.0 (c) 0.5 (d) 3.0

Question 17: Using data points: $(0,1), (0.1,1.1), (0.2,1.3), (0.3,1.4), (0.4,1.5), (0.5,1.7)$, then the **best** approximate value of $f'(0)$ using 3-point difference formula is:

(a) 0.5 (b) 0.75 (c) 0.55 (d) 0.6

Question 18: Using data points: $(0,1), (0.1,1.1), (0.2,1.3), (0.3,1.4), (0.4,1.5), (0.5,1.7)$, then the **worst** approximate value of $f'(0.25)$ using 3-point difference formula is:

(a) 1.4 (b) 1.45 (c) 1.5 (d) 1.55

Question 19: Using data points: $(0,1), (0.1,1.1), (0.2,1.3), (0.3,1.4), (0.4,1.5), (0.5,1.7)$, then the **worst** approximate value of $f'(0.4)$ using 3-point difference formula is:

(a) 0.75 (b) 1.75 (c) 0.95 (d) 1.25

Question 20: Given initial-value problem $y' = x + y,\ y(0) = 1$, the approximate value of $y(0.1)$ using Euler's method with $n = 1$ is:
(a) 1.1 (b) 1.01 (c) 1.02 (d) 1.2

Question 21: Consider the following system of linear equations [5 points]

$$\begin{array}{rcrcrcl} 9x_1 & + & 3x_2 & - & x_3 & = & 1 \\ 4x_1 & + & 10x_2 & - & 3x_3 & = & 1 \\ 2x_1 & + & 3x_2 & + & 9x_3 & = & 1 \end{array}$$

If the initial approximation is $\mathbf{x^{(0)}} = [0.05, 0.05, 0.05]^T$, then find the second approximation $\mathbf{x^{(2)}}$ using Jacobi iterative method and compute an error bound for your approximation. Also, compute the number of iterations required to approximate the solution within accuracy 10^{-4}.

Solution. The Jacobi iterative method for the given system is

$$x_1^{(k+1)} = \frac{1}{9}\Big[1 - 3x_2^{(k)} + x_3^{(k)}\Big]$$

$$x_2^{(k+1)} = \frac{1}{10}\Big[1 - 4x_1^{(k)} + 3x_3^{(k)}\Big]$$

$$x_3^{(k+1)} = \frac{1}{9}\Big[1 - 2x_1^{(k)} - 3x_2^{(k)}\Big]$$

and starting with initial approximation $x_1^{(0)} = 0.05, x_2^{(0)} = 0.05, x_3^{(0)} = 0.05$, then for $k = 0, 1$: $\mathbf{x^{(1)}} = [0.1000, 0.0950, 0.0833]^T$ and $\mathbf{x^{(2)}} = [0.0887, 0.0850, 0.0572]^T$.
The Jacobi iteration matrix is defined as

$$T_J = -D^{-1}(L+U) = \begin{pmatrix} 0 & -\frac{1}{3} & \frac{1}{9} \\ -\frac{4}{10} & 0 & \frac{3}{10} \\ -\frac{2}{9} & -\frac{3}{9} & 0 \end{pmatrix}, \quad \|T_J\|_\infty = \max\left\{\frac{4}{9}, \frac{7}{10}, \frac{5}{9}\right\} = \frac{7}{10} < 1$$

Using the error bound formula, we obtain

$$\|\mathbf{x} - \mathbf{x^{(2)}}\| \leq \frac{(7/10)^2}{1 - 7/10}\left\|\begin{pmatrix} 0.1000 \\ 0.0950 \\ 0.0833 \end{pmatrix} - \begin{pmatrix} 0.05 \\ 0.05 \\ 0.05 \end{pmatrix}\right\| = 0.0817$$

To find the number of iterations, we do the following

$$\|\mathbf{x} - \mathbf{x^{(k)}}\| \leq \frac{\|T_J\|^k}{1 - \|T_J\|}\|\mathbf{x}^{(1)} - \mathbf{x^{(0)}}\| \leq 10^{-4}$$

It gives

$$\frac{(0.7)^k}{0.3}(0.05) \leq 10^{-4}, \text{ or } (0.7)^k \leq \frac{10^{-4}}{0.1667}$$

Taking ln on both sides, we obtain

$$k\ln(0.7) \leq \ln(0.0006) \text{ gives } k \geq 20.7998 \quad \text{or} \quad k = 21.$$

Question 22: Given $f(x) = x^{1/3}$, and $x_0 = 0, x_1 = 1, x_2 = 27, x_3 = 64$. Construct the divided differences table for the function. Find the linear spline which interpolates this data. Find approximation of $f(8)$ and the absolute error. [5 points]

Solution: The divided differences table for the given function $f(x) = x^{1/3}$, and at the points $x_0 = 0, x_1 = 1, x_2 = 27, x_3 = 64$ is as follows:

k	x_k	Zeroth Divided Difference	First Divided Difference	Second Divided Difference	Third Divided Difference
0	$x_0 = 0$	$f[x_0] = 0$			
1	$x_1 = 1$	$f[x_1] = 1$	$f[x_0, x_1] = 1$		
2	$x_2 = 27$	$f[x_2] = 3$	$f[x_1, x_2] = 0.0769$	$f[x_0, x_1, x_2] = -0.0342$	
3	$x_3 = 64$	$f[x_3] = 4$	$f[x_2, x_3] = 0.0270$	$f[x_1, x_2, x_3] = -0.0008$	$f[x_0, x_1, x_2, x_3] = 0.0005$

Linear spline:

$$s_k(x) = A_k + B_k(x - x_k)$$

where the values of the coefficients A_k and B_k are given as

$$A_k = y_k \quad \text{and} \quad B_k = f[x_k, x_{k+1}] = \frac{(y_{k+1} - y_k)}{(x_{k+1} - x_k)}$$

Given $x_0 = 0, x_1 = 1, x_2 = 27, x_3 = 64$, then we have

$$\begin{aligned} A_0 &= y_0 = 0 \\ A_1 &= y_1 = 1 \\ A_2 &= y_2 = 3 \\ A_3 &= y_3 = 4 \end{aligned}$$

and

$$\begin{aligned} B_0 &= f[x_0, x_1] = 1 \\ B_1 &= f[x_1, x_2] = 0.0769 \\ B_2 &= f[x_2, x_3] = 0.0270 \end{aligned}$$

Now the linear spline for three subintervals are define as

$$s(x) = \begin{cases} s_0(x) = A_0 + B_0(x - x_0) = 0 + 1(x - 0) = x, & 0 \leq x \leq 1 \\ s_1(x) = A_1 + B_1(x - x_1) = 1 + 0.0769(x - 1) = 0.9231 + 0.0769x, & 1 \leq x \leq 27 \\ s_2(x) = A_2 + B_2(x - x_2) = 3 + 0.0270(x - 27) = 2.2710 + 0.0270x, & 27 \leq x \leq 64 \end{cases}$$

The value $x = 8$ lies in the interval $[1, 27]$, so

$$f(8) \approx s_1(8) = 0.9231 + 0.0769(8) = 1.5383$$

Absolute error $= |f(8) - s_1(8)| = |2 - 1.5383| = 0.4615.$

Question 23: **(i)** Let $f(x) \in C^2[x_0, x_1]$ and $h = x_1 - x_0$, then show that

$$\int_{x_0}^{x_1} f(x)\, dx \approx \frac{h}{2}[f(x_0) + f(x_1)]$$

(ii) Use the best numerical integration rule to approximate $\int_0^2 \frac{1}{1+2x}\, dx$ when $h = 0.5$. Also, compute an error bound for your approximation.

[5 points]

Solution: **(i)** Consider the first degree Lagrange interpolating polynomial with equally spaced data points, that is, x_0 and x_1 with $h = x_1 - x_0$, then

$$f(x) \approx p_1(x) = \left(\frac{x - x_1}{x_0 - x_1}\right) f(x_0) + \left(\frac{x - x_0}{x_1 - x_0}\right) f(x_1) \tag{1}$$

Taking integral on both sides of (**??**) with respect to x between the limits x_0 and x_1, we have

$$\int_{x_0}^{x_1} f(x)dx \approx \frac{f(x_0)}{x_0 - x_1} \int_{x_0}^{x_1} (x - x_1)dx + \frac{f(x_1)}{x_1 - x_0} \int_{x_0}^{x_1} (x - x_0)dx$$

$$\int_{x_0}^{x_1} f(x)dx \approx \frac{f(x_0)}{x_0 - x_1} \left[(x - x_1)^2/2\Big|_{x_0}^{x_1}\right] + \frac{f(x_1)}{x_1 - x_0} \left[(x - x_0)^2/2\Big|_{x_0}^{x_1}\right]$$

$$\int_{x_0}^{x_1} f(x)dx \approx \frac{f(x_0)}{x_0 - x_1} \left[0 - (x_0 - x_1)^2/2\right] + \frac{f(x_1)}{x_1 - x_0} \left[(x_1 - x_0)^2/2 - 0\right]$$

$$\int_{x_0}^{x_1} f(x)dx \approx \frac{(x_1 - x_0)}{2} \Big[f(x_0) + f(x_1)\Big] = \frac{h}{2}\Big[f(x_0) + f(x_1)\Big].$$

(ii) Since $h = 0.5$, gives $n = (2 - 0)/0.5 = 4$, so best numerical integration rule is Simpson's rule and for $n = 4$, we have Simpson's rule is

$$\int_{x_0}^{x_4} f(x)dx \approx \frac{h}{3}[f(x_0) + 4(f(x_1) + f(x_3)) + 2f(x_2) + f(x_4)]$$

Since

$$\int_0^2 \frac{1}{1+2x}\, dx \approx \frac{0.5}{3}[f(0) + 4(f(0.5) + f(1.5)) + 2f(1) + f(2)]$$

gives

$$\int_0^2 \frac{1}{1+2x}\, dx \approx \frac{0.5}{3}[1 + 4(3/4) + 2(1/3) + 1/6] = 29/36 = 0.8056.$$

The error bound for the approximation is

$$|E_{T_4}(f)| \leq \frac{h^4(b-a)}{180} M$$

$$|f^{(4)}| \leq M = \max_{0 \leq x \leq 2} |f^{(4)}(x)| = \max_{0 \leq x \leq 2} |384/(1+2x)^5| = 384$$

$$|E_{T_4}(f)| \leq \frac{(0.5)^4(2)}{180}(384) = 0.2667.$$

Question 24: Consider the initial-value problem [5 points]

$$y' - y = x^2, \qquad y(0) = 1.$$

Use the Runge-Kutta method of order two (Modified Euler's method) with $n = 2$ to compute approximation for $y(0.4)$.

Solution: Runge-Kutta method of order two (Modified Euler's method) is

$$y_{i+1} = y_i + \frac{h}{2}[k_1 + k_2]$$

where

$$\begin{aligned} k_1 &= f(x_i, y_i) \\ k_2 &= f(x_{i+1}, y_i + hk_1) \end{aligned}$$

for each $i = 0, 1, \dots n-1$. Since $f(x,y) = x^2 + y,\ x_0 = 0,\ y_0 = 1$, and $n = 2$ gives $h = 0.2$. Then for $i = 0$, we have

$$\begin{aligned} k_1 &= f(x_0, y_0) = (x_0^2 + y_0) = (0 + 1) = 1 \\ k_2 &= f(x_1, y_0 + hk_1) = f(0.2, 1 + 0.2(1)) = 0.2^2 + 1.2 = 1.24 \end{aligned}$$

and using these values, we have

$$y_1 = y_0 + \frac{h}{2}\,[k_1 + k_2] = 1 + 0.1(1 + 1.24) = 1.224.$$

Similarly, for $i = 1$, we have

$$\begin{aligned} k_1 &= f(x_1, y_1) = (x_1^2 + y_1) = (0.04 + 1.224) = 1.264 \\ k_2 &= f(x_2, y_1 + hk_1) = f(0.4, 1.224 + 0.2(1.264)) == f(0.4, 1.4768) = 0.4^2 + 1.4768 = 1.6368 \end{aligned}$$

and using these values, we have

$$y_2 = y_1 + \frac{h}{2}\,[k_1 + k_2] = 1.224 + 0.1(1.264 + 1.6368) = 1.5141.$$

Thus

$$\begin{aligned} x_0 &= 0.0 \quad \text{given} \quad y_0 = 1 \\ x_1 &= 0.2 \quad \text{gives} \quad y_1 = 1.224 \\ x_2 &= 0.4 \quad \text{gives} \quad y_2 = 1.5141 \end{aligned}$$