King Saud University: Mathematics Department Math-254
First Semester 1439-40 H Final Examination
Maximum Marks = 40 Time: 180 mins.

Name of the Student:———————— I.D. No. ————

Name of the Teacher:———————— Section No. ————

Note: Check the total number of pages are Seven (7).

(10 Multiple choice questions and Four (4) Full questions)

The Answer Tables for Q.1 to Q.10 : Marks: 2 for each one ($2 \times 10 = 20$)

Ps. : Mark {a, b, c or d} for the correct answer in the box.

Q. No.	1	2	3	4	5	6	7	8	9	10
a,b,c,d										

Quest. No.	Marks Obtained	Marks for Question
Q. 1 to Q. 10		20
Q. 11		5
Q. 12		5
Q. 13		5
Q. 14		5
Total		40

Question 1: The value of c which insures quadratic convergence of $x_{n+1} = x_n + c(x_n^2 - 3)$, to $\alpha = \sqrt{3}$ is:

(a) $-\frac{1}{2\sqrt{3}}$ (b) $\frac{1}{2\sqrt{3}}$ (c) $-\frac{1}{\sqrt{3}}$ (d) Non of these

Question 2: If $x_{n+1} = g(x_n) = \ln(x_n + 2)$, $x_0 = 1.5$ and $k = max|g'(x)| = \frac{1}{3}$, then the number of iterations to achieve accuracy 10^{-2} is:

(a) 2 (b) 3 (c) 4 (d) Non of these

Question 3: The second approximation of the square root of 19 using a quadratic convergent method when $x_0 = 5$ is:

(a) 4.4 (b) 4.44 (c) 4.359 (d) Non of these

Question 4: The order of convergence of $x_{n+1} = 2x_n^2 + \frac{4}{x_n} - 5$, $n \geq 0$, to $\alpha = 1$ is:

(a) *linear* (b) *Atleast quadratic* (c) *quadratic* (d) Non of these

Question 5: The error bound of $\|x - x^{(5)}\|$ using Jacobi iterative method with $x^{(0)} = (0,0)^T$, for solving linear system $A\mathbf{x} = \mathbf{b}$, where $A = \begin{pmatrix} 4 & -1 \\ -1 & 3 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 12 \\ 1 \end{pmatrix}$ is:

(a) 0.01851 (b) 0.0039 (c) 0.0205 (d) Non of these

Question 6: If $f(x) = \frac{2}{x}$, then $f[1,2,1]$ is equal to:

(a) 0 (b) 1 (c) 5 (d) -3

Question 7: If $x_0 = 1, x_1 = 2$, $x_2 = 3$, $x_3 = 4$ and $f(x) = \frac{2}{x+1}$, then the absolute error of approximating $f(2.9)$ using linear spline function is:

(a) 0.004 (b) 0.14 (c) 0.03 (d) Non of these

Question 8: The best approximation of $f'(1.5)$ using three point difference formula for the function $f(x) = \ln x$ and $h = 0.5$ is:

(a) 0.6931 (b) 0.6399 (c) 0.5232 (d) Non of these

Question 9: The error bound of approximating the integral $\int_1^2 \frac{1}{x+1}\,dx$, using simple Trapezoidal rule is:

(a) 0.0208 (b) 0.00617 (c) 0.0833 (d) Non of these

Question 10: Given $\frac{y'}{x} - y^2 = 0$, $y(1.2) = 1.1$, the approximate value of $y(1.4)$ using Taylor's method of order 1 (Euler's method) when $n = 1$ is:

(a) 1.245 (b) 1.545 (c) 1.582 (d) Non of these

Question 11: Consider a linear system $\boxed{A\mathbf{x} = \mathbf{b},}$ where [5 points]

$$A = \begin{pmatrix} 2 & 1 & 2 \\ 1 & 4 & 0 \\ 1 & 2 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}.$$

Discuss the conditioning of the given linear system. Suppose that $\boxed{\mathbf{b} = A\mathbf{x}}$ is changed to $\boxed{\mathbf{b}^* = A\mathbf{x}^*} = [1, 1, 1.99]^T$. How large a relative change can this change produce in the solution to $A\mathbf{x} = \mathbf{b}$?

Solution. Since the matrix A and its inverse is

$$A = \begin{pmatrix} 2 & 1 & 2 \\ 1 & 4 & 0 \\ 1 & 2 & 1 \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} 4/3 & 1 & -8/3 \\ -1/3 & 0 & 2/3 \\ -2/3 & -1 & 7/3 \end{pmatrix}.$$

Then

$$\|A\|_\infty = 5, \|A^{-1}\|_\infty = 5, \qquad K(A) = \|A\|_\infty \|A^{-1}\|_\infty = (5)(5) = 25.$$

Since the change from $\mathbf{b}$ to $\mathbf{b}^*$ is an error $\delta\mathbf{b}$, that is, $\mathbf{b}^* = \mathbf{b} + \delta\mathbf{b}$, so

$$\delta\mathbf{b} = \begin{pmatrix} -0.01 \\ 0 \\ 0 \end{pmatrix} = -\mathbf{r},$$

and the l_∞-norm of this column matrix is, $\|\delta\mathbf{b}\|_\infty = 0.01$. From the equation (??), we get

$$\frac{\|\delta\mathbf{x}\|}{\|\mathbf{x}\|} \leq \frac{25(0.01)}{2} = 0.1250,$$

the possible relative change in the solution to the given linear system. •

Table 1: Divide differences table for the Example ??.

k	x_k	Zeroth Divided Difference	First Divided Difference	Second Divided Difference	Third Divided Difference
0	0	0.6932			
1	1	1.0986	0.4055		
2	2	1.3863	0.2877	- 0.0589	
3	3	1.6094	0.2232	- 0.0323	0.0089

Question 12: Construct the divided difference table for the function $f(x) = \ln(x+2)$ in the interval $0 \leq x \leq 3$ for the stepsize $h = 1$. Find second degree Newton divided difference interpolating polynomial to construct the interpolating polynomial degree 3 for the approximation of $\ln(3.5)$. Compute error bound $\|f(x) - p_3(x)\|$. [5 points]

Solution. The results of the divided differences are listed in Table 1.

Firstly, we construct the second degree polynomial $p_2(x)$ by using the quadratic Newton interpolation formula as follows

$$f(x) = p_2(x) = f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1),$$

then with the help of the divided differences Table 1, we get

$$f(x) = p_2(x) = 0.6932 + 0.4055(x - 0) - 0.0589(x - 0)(x - 1),$$

which implies that

$$f(x) = p_2(x) = -0.0568x^2 + 0.4644x + 0.6932 \quad \text{and} \quad p_2(1.5) = 1.2620.$$

Now to construct the cubic interpolatory polynomial $p_3(x)$ that fits at all four points. We only have to add one more term to the polynomial $p_2(x)$:

$$f(x) = p_3(x) = p_2(x) + f[x_0, x_1, x_2, x_3](x - x_0)(x - x_1)(x - x_2),$$

and this gives

$$f(x) = p_3(x) = p_2(x) + 0.0089(x^3 - 3x^2 + 2x) \text{ and } f(1.5) \approx p_3(1.5) = 1.2620 - 0.0033 = 1.2587.$$

We note that the estimated value of $f(1.5)$ by cubic interpolating polynomial is more closer to the exact solution than the quadratic polynomial.

(c) Now to compute the error bound for the approximation $p_3(x)$, we use the error formula

$$|f(x) - p_3(x)| = \frac{|f^{(4)}(\eta(x))|}{4!}|(x - x_0)(x - x_1)(x - x_2)(x - x_3)|.$$

Taking the fourth derivative of the given function, we obtain

$$f^{(4)}(x) = \frac{-6}{(x+2)^4} \quad \text{and} \quad |f^{(4)}(\eta(x))| = \left|\frac{-6}{(\eta(x)+2)^4}\right|, \quad \text{for} \quad \eta(x) \in (0, 3).$$

Since

$$|f^{(4)}(0)| = 0.375 \quad \text{and} \quad |f^{(4)}(3)| = 0.0096,$$

so $|f^{(4)}(\eta(x))| \leq \max_{0 \leq x \leq 3} \left| \dfrac{-6}{(x+2)^4} \right| = 0.375$ and it gives

$$|f(1.5) - p_3(1.5)| \leq (0.5625)(0.375)/24 = 0.0088,$$

which is the required error bound for the approximation $p_3(1.5)$. •

Question 13: Find the approximation of $f''(0.8)$ by using the following set of data points using numerical rule:

x	0.0	0.11	0.24	0.3	0.4	0.5	0.6	0.72	0.8	0.9	1.05	1.11	1.2
$f(x)$	1.00	1.10	1.2	1.26	1.32	1.38	1.43	1.47	1.50	1.52	1.55	1.55	1.56

If the function is $f(x) = x + \cos x$, then compute an error bound for your approximation. How many subintervals approximate the given integral to within accuracy of 10^{-6} using this differentiation rule ? [5 points]

Solution. Given $x_1 = 0.8, h = 0.4$, then the formula (??) becomes

$$f''(1) \approx \frac{f(0.8+0.4) - 2f(0.8) + f(0.8-0.4)}{(0.4)^2} = D_h^2 f(1),$$

or

$$f''(1) \approx \frac{f(1.2) - 2f(0.8) + f(0.4)}{0.16} = -0.75 = D_h^2 f(1).$$

To compute the error bound for our approximation in part (a), we use the formula as

$$|E_C(f,h)| = \Big| - \frac{h^2}{12} \Big| |f^{(4)}(\eta(x_1))|, \quad \text{for} \quad \eta(x_1) \in (0.9, 1.1).$$

The fourth derivative of the given function at $\eta(x_1)$ is $f^{(4)}(\eta(x_1)) = \cos \eta(x_1)$, and it cannot be computed exactly because $\eta(x_1)$ is not known. But one can bound the error by computing the largest possible value for $|f^{(4)}(\eta(x_1))|$. So bound $|f^{(4)}|$ on the interval $(0.9, 1.1)$ is

$$M = \max_{0.4 \le x \le 1.2} |\cos x)| = 0.921061,$$

at $x = 1.1$. Thus, for $|f^{(4)}(\eta(x))| \le M$, we have the possible maximum error as

$$|E_C(f,h)| \le \frac{h^2}{12} M \le \frac{(0.1)^2}{12}(0.4536) = 0.000767.$$

(d) Since the given accuracy required is 10^{-2}, so

$$|E_C(f,h)| = \Big| - \frac{h^2}{12} f^{(4)}(\eta(x_1)) \Big| \le 10^{-2},$$

for $\eta(x_1) \in (0.4, 1.2)$. Then for $|f^{(4)}(\eta(x_1))| \le M$, we have

$$\frac{h^2}{12} M \le 10^{-2}, \quad h^2 \le \frac{(12 \times 10^{-2})}{M} = \frac{(12 \times 10^{-2})}{0.92106} = 0.013, \quad h \le 0.3609.$$

Question 14: Determine the number of subintervals n required to approximate

$$I(f) = \int_0^2 \frac{1}{x+4}\,dx,$$

with an error less than 10^{-4} using Simpson's rule. Then approximate the given integral. Find the absolute error. [5 points]

Solution. we have to use the error formula (??) which is

$$|E_{S_n}(f)| \leq \frac{(b-a)}{180} h^4 M \leq 10^{-4}.$$

Given the integrand is $f(x) = \frac{1}{x+4}$, and we have $f^{(4)}(x) = \frac{24}{(x+4)^5}$. The maximum value of $|f^{(4)}(x)|$ on the interval $[0, 2]$ is $3/128$, and thus $M = \frac{3}{128}$. Using the above error formula, we get

$$\frac{3}{(90 \times 128)} h^4 \leq 10^{-4}, \quad \text{or} \quad h \leq \frac{2}{5}\sqrt[4]{15} = 0.7872.$$

Since $n = \frac{2}{h} = \frac{2}{0.7872} = 2.5407$, so the number of even subintervals n required is $n \geq 4$. Thus the approximation of the given integral using $h = \frac{2-0}{4} = \frac{1}{2} = 0.5$ is

$$\int_0^2 \frac{1}{x+4} \approx \frac{0.5}{3}\Big[f(0) + 4[f(0.5) + f(1.5)] + 2f(1) + f(2)\Big],$$

$$\int_0^1 \frac{1}{x+4} \approx \frac{1}{6}\Big[0.25 + 4(0.2222 + 0.1818) + 2(0.2) + 0.1667\Big] = 0.4055,$$

which is equal to the true value of the given integral $\alpha = \ln(1.5) = 0.4055$ upto 4 decimal places.

The Answer Table for Q.1 to Q.10 : Math

Check the correct answer in the box.

Q. No.	1	2	3	4	5	6	7	8	9	10
a,b,c,d	a	c	c	c	a	b	a	b	c	a

The Answer Table for Q.1 to Q.10 : MATH

Check the correct answer in the box.

Q. No.	1	2	3	4	5	6	7	8	9	10
a,b,c,d	b	a	a	b	c	c	b	a	b	c