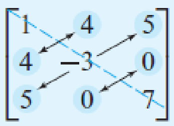
Symmetric Matrices

DEFINITION 1 A square matrix A is said to be ***symmetric*** if $A = A^T$.

It is easy to recognize a symmetric matrix by inspection: The entries on the main diagonal have no restrictions, but mirror images of entries *across* the main diagonal must be equal. Here is a picture using the second matrix in Example 4:



EXAMPLE 4 Symmetric Matrices

The following matrices are symmetric, since each is equal to its own transpose (verify).

$$\begin{bmatrix} 7 & -3 \\ -3 & 5 \end{bmatrix}, \quad \begin{bmatrix} 1 & 4 & 5 \\ 4 & -3 & 0 \\ 5 & 0 & 7 \end{bmatrix}, \quad \begin{bmatrix} d_1 & 0 & 0 & 0 \\ 0 & d_2 & 0 & 0 \\ 0 & 0 & d_3 & 0 \\ 0 & 0 & 0 & d_4 \end{bmatrix}$$

Remark It follows from Formula (14) of Section 1.3 that a square matrix A is symmetric if and only if

$$(A)_{ij} = (A)_{ji} \tag{4}$$

for all values of i and j.

Remark.

a matrix A is said to be ***skew-symmetric*** if $A^T = -A$.

Symmetric	Skew-symmetric
$A^T = A$	$A^T = -A$
$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 0 \\ -1 & 0 & 5 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix}$

THEOREM 1.7.2 *If A and B are symmetric matrices with the same size, and if k is any scalar, then:*

(*a*) A^T *is symmetric.*

(*b*) $A + B$ *and* $A - B$ *are symmetric.*

(*c*) kA *is symmetric.*

It is not true, in general, that the product of symmetric matrices is symmetric. To see why this is so, let A and B be symmetric matrices with the same size. Then it follows from part (*e*) of Theorem 1.4.8 and the symmetry of A and B that

$$(AB)^T = B^T A^T = BA$$

Thus, $(AB)^T = AB$ if and only if $AB = BA$, that is, if and only if A and B commute. In summary, we have the following result.

THEOREM 1.7.3 *The product of two symmetric matrices is symmetric if and only if the matrices commute.*

Invertibility of Symmetric Matrices

In general, a symmetric matrix need not be invertible. For example, a diagonal matrix with a zero on the main diagonal is symmetric but not invertible. However, the following theorem shows that if a symmetric matrix happens to be invertible, then its inverse must also be symmetric.

THEOREM 1.7.4 *If A is an invertible symmetric matrix, then A^{-1} is symmetric.*

2.1 Determinants by Cofactor Expansion

Minors and Cofactors One of our main goals in this chapter is to obtain an analog of Formula (2) that is applicable to square matrices of *all orders*. For this purpose we will find it convenient to use subscripted entries when writing matrices or determinants. Thus, if we denote a 2×2 matrix as

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

then the two equations in (1) take the form

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21} \tag{3}$$

There are various methods for defining determinants of higher-order square matrices. In this text, we will us an "inductive definition" by which we mean that the determinant of a square matrix of a given order will be defined in terms of determinants of square matrices of the next lower order. To start the process, let us define the determinant of a 1×1 matrix $[a_{11}]$ as

$$\det[a_{11}] = a_{11} \tag{5}$$

DEFINITION 1 If A is a square matrix, then the ***minor of entry*** a_{ij} is denoted by M_{ij} and is defined to be the determinant of the submatrix that remains after the ith row and jth column are deleted from A. The number $(-1)^{i+j}M_{ij}$ is denoted by C_{ij} and is called the ***cofactor of entry*** a_{ij}.

▶ EXAMPLE 1 Finding Minors and Cofactors

Let

$$A = \begin{bmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{bmatrix}$$

The minor of entry a_{11} is

$$M_{11} = \begin{vmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{vmatrix} = \begin{vmatrix} 5 & 6 \\ 4 & 8 \end{vmatrix} = 16$$

The cofactor of a_{11} is

$$C_{11} = (-1)^{1+1}M_{11} = M_{11} = 16$$

Remark Note that a minor M_{ij} and its corresponding cofactor C_{ij} are either the same or negatives of each other and that the relating sign $(-1)^{i+j}$ is either $+1$ or -1 in accordance with the pattern in the "checkerboard" array

$$\begin{bmatrix} + & - & + & - & + & \cdots \\ - & + & - & + & - & \cdots \\ + & - & + & - & + & \cdots \\ - & + & - & + & - & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

For example,

$$C_{11} = M_{11}, \quad C_{21} = -M_{21}, \quad C_{22} = M_{22}$$

and so forth. Thus, it is never really necessary to calculate $(-1)^{i+j}$ to calculate C_{ij}—you can simply compute the minor M_{ij} and then adjust the sign in accordance with the checkerboard pattern. Try this in Example 1.

DEFINITION 2 If A is an $n \times n$ matrix, then the number obtained by multiplying the entries in any row or column of A by the corresponding cofactors and adding the resulting products is called the ***determinant of*** A, and the sums themselves are called ***cofactor expansions of*** A. That is,

$$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj} \tag{7}$$

[cofactor expansion along the jth column]

and

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in} \tag{8}$$

[cofactor expansion along the ith row]

EXAMPLE 5 Smart Choice of Row or Column

If A is the 4×4 matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 3 & 1 & 2 & 2 \\ 1 & 0 & -2 & 1 \\ 2 & 0 & 0 & 1 \end{bmatrix}$$

then to find $\det(A)$ it will be easiest to use cofactor expansion along the second column, since it has the most zeros:

$$\det(A) = 1 \cdot \begin{vmatrix} 1 & 0 & -1 \\ 1 & -2 & 1 \\ 2 & 0 & 1 \end{vmatrix}$$

For the 3×3 determinant, it will be easiest to use cofactor expansion along its second column, since it has the most zeros:

$$\begin{aligned} \det(A) &= 1 \cdot -2 \cdot \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} \\ &= -2(1+2) \\ &= -6 \end{aligned}$$

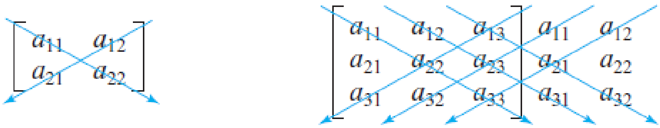
▶ EXAMPLE 6 Determinant of a Lower Triangular Matrix

The following theorem [redacted] shows that the determinant of a 4×4 lower triangular matrix is the product of its diagonal entries.

> **THEOREM 2.1.2** *If A is an $n \times n$ triangular matrix (upper triangular, lower triangular, or diagonal), then $\det(A)$ is the product of the entries on the main diagonal of the matrix; that is,* $\det(A) = a_{11}a_{22}\cdots a_{nn}$.

A Useful Technique for Evaluating 2×2 and 3×3 Determinants

Determinants of 2×2 and 3×3 matrices can be evaluated very efficiently using the pattern suggested in Figure 2.1.1.



▶ Figure 2.1.1

▶ EXAMPLE 7 A Technique for Evaluating 2 x 2 and 3 x 3 Determinants

$$\begin{vmatrix} 3 & 1 \\ 4 & -2 \end{vmatrix} = \begin{vmatrix} 3 & 1 \\ 4 & -2 \end{vmatrix} = (3)(-2) - (1)(4) = -10$$

$$\begin{vmatrix} 1 & 2 & 3 \\ -4 & 5 & 6 \\ 7 & -8 & 9 \end{vmatrix} = \left|\begin{matrix} 1 & 2 & 3 \\ -4 & 5 & 6 \\ 7 & -8 & 9 \end{matrix}\right|\begin{matrix} 1 & 2 \\ -4 & 5 \\ 7 & -8 \end{matrix}$$

$$= [45 + 84 + 96] - [105 - 48 - 72] = 240 \blacktriangleleft$$

2.2 Evaluating Determinants by Row Reduction

THEOREM 2.2.1 *Let A be a square matrix. If A has a row of zeros or a column of zeros, then* $\det(A) = 0$.

THEOREM 2.2.2 *Let A be a square matrix. Then* $\det(A) = \det(A^T)$.

Elementary Row Operations

The next theorem shows how an elementary row operation on a square matrix affects the value of its determinant. In place of a formal proof we have provided a table to illustrate the ideas in the 3×3 case (see Table 1).

THEOREM 2.2.3 *Let A be an $n \times n$ matrix.*

(a) *If B is the matrix that results when a single row or single column of A is multiplied by a scalar k, then* $\det(B) = k\det(A)$.

(b) *If B is the matrix that results when two rows or two columns of A are interchanged, then* $\det(B) = -\det(A)$.

(c) *If B is the matrix that results when a multiple of one row of A is added to another or when a multiple of one column is added to another, then* $\det(B) = \det(A)$.

Table 1

Relationship	Operation
$\begin{vmatrix} ka_{11} & ka_{12} & ka_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = k \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ $\det(B) = k\det(A)$	In the matrix B the first row of A was multiplied by k.
$\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = - \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ $\det(B) = -\det(A)$	In the matrix B the first and second rows of A were interchanged.
$\begin{vmatrix} a_{11}+ka_{21} & a_{12}+ka_{22} & a_{13}+ka_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ $\det(B) = \det(A)$	In the matrix B a multiple of the second row of A was added to the first row.

Matrices with Proportional Rows or Columns

If a square matrix A has two proportional rows, then a row of zeros can be introduced by adding a suitable multiple of one of the rows to the other. Similarly for columns. But adding a multiple of one row or column to another does not change the determinant, so from Theorem 2.2.1, we must have $\det(A) = 0$. This proves the following theorem.

THEOREM 2.2.5 *If A is a square matrix with two proportional rows or two proportional columns, then* $\det(A) = 0$.

EXAMPLE 2 Proportional Rows or Columns

Each of the following matrices has two proportional rows or columns; thus, each has a determinant of zero.

$$\begin{bmatrix} -1 & 4 \\ -2 & 8 \end{bmatrix}, \quad \begin{bmatrix} 1 & -2 & 7 \\ -4 & 8 & 5 \\ 2 & -4 & 3 \end{bmatrix}, \quad \begin{bmatrix} 3 & -1 & 4 & -5 \\ 6 & -2 & 5 & 2 \\ 5 & 8 & 1 & 4 \\ -9 & 3 & -12 & 15 \end{bmatrix}$$

Evaluating Determinants by Row Reduction

We will now give a method for evaluating determinants that involves substantially less computation than cofactor expansion. The idea of the method is to reduce the given matrix to upper triangular form by elementary row operations, then compute the determinant of the upper triangular matrix (an easy computation), and then relate that determinant to that of the original matrix. Here is an example.

> Even with today's fastest computers it would take millions of years to calculate a 25×25 determinant by cofactor expansion, so methods based on row reduction are often used for large determinants. For determinants of small size (such as those in this text), cofactor expansion is often a reasonable choice.

▶ EXAMPLE 3 Using Row Reduction to Evaluate a Determinant

Evaluate $\det(A)$ where

$$A = \begin{bmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{bmatrix}$$

Solution We will reduce A to row echelon form (which is upper triangular) and then apply Theorem 2.1.2.

$$\det(A) = \begin{vmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{vmatrix} = -\begin{vmatrix} 3 & -6 & 9 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix}$$

← The first and second rows of A were interchanged.

$$= -3\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix}$$

← A common factor of 3 from the first row was taken through the determinant sign.

$$= -3\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 10 & -5 \end{vmatrix}$$

← -2 times the first row was added to the third row.

$$= -3\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & -55 \end{vmatrix}$$

← -10 times the second row was added to the third row.

$$= (-3)(-55)\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{vmatrix}$$

← A common factor of -55 from the last row was taken through the determinant sign.

$$= (-3)(-55)(1) = 165$$

EXAMPLE 4 Using Column Operations to Evaluate a Determinant

Compute the determinant of

$$A = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 2 & 7 & 0 & 6 \\ 0 & 6 & 3 & 0 \\ 7 & 3 & 1 & -5 \end{bmatrix}$$

Solution This determinant could be computed as above by using elementary row operations to reduce A to row echelon form, but we can put A in lower triangular form in one step by adding -3 times the first column to the fourth to obtain

$$\det(A) = \det \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 7 & 0 & 0 \\ 0 & 6 & 3 & 0 \\ 7 & 3 & 1 & -26 \end{bmatrix} = (1)(7)(3)(-26) = -546$$

> Example 4 points out that it is always wise to keep an eye open for column operations that can shorten computations.

Cofactor expansion and row or column operations can sometimes be used in combination to provide an effective method for evaluating determinants. The following example illustrates this idea.

Cofactor expansion and row or column operations can sometimes be used in combination to provide an effective method for evaluating determinants. The following example illustrates this idea.

EXAMPLE 5 Row Operations and Cofactor Expansion

Evaluate $\det(A)$ where

$$A = \begin{bmatrix} 3 & 5 & -2 & 6 \\ 1 & 2 & -1 & 1 \\ 2 & 4 & 1 & 5 \\ 3 & 7 & 5 & 3 \end{bmatrix}$$

Solution By adding suitable multiples of the second row to the remaining rows, we obtain

$$\det(A) = \begin{vmatrix} 0 & -1 & 1 & 3 \\ 1 & 2 & -1 & 1 \\ 0 & 0 & 3 & 3 \\ 0 & 1 & 8 & 0 \end{vmatrix}$$

$$= -\begin{vmatrix} -1 & 1 & 3 \\ 0 & 3 & 3 \\ 1 & 8 & 0 \end{vmatrix} \quad \leftarrow \text{Cofactor expansion along the first column}$$

$$= -\begin{vmatrix} -1 & 1 & 3 \\ 0 & 3 & 3 \\ 0 & 9 & 3 \end{vmatrix} \quad \leftarrow \text{We added the first row to the third row.}$$

$$= -(-1)\begin{vmatrix} 3 & 3 \\ 9 & 3 \end{vmatrix} \quad \leftarrow \text{Cofactor expansion along the first column}$$

$$= -18$$