> **DEFINITION 1** Matrices A and B are said to be ***row equivalent*** if either (hence each) can be obtained from the other by a sequence of elementary row operations.

Example.

Show that the matrix $\mathbf{A} = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & -1 & 3 \end{bmatrix}$ is row equivalent to the matrix $\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

$$\begin{pmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & -1 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 2 \\ 0 & -1 & -1 \\ 0 & -3 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & +2 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Homework.

Which of the following matrices is row equivalent to the matrix $\begin{bmatrix} 1 & 1 & 1 \\ x & y & z \\ z+y & x+z & y+x \end{bmatrix}$?

(a) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ y+z & x+z & x+y \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 1 & z \\ 1 & 1 & y \\ 1 & 1 & x \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 1 & 1 \\ x & y & z \\ z & y & x \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ x+z & 2y & x+z \end{bmatrix}$.

Inverse of a Matrix

DEFINITION 1 If A is a square matrix, and if a matrix B of the same size can be found such that $AB = BA = I$, then A is said to be ***invertible*** (or ***nonsingular***) and B is called an ***inverse*** of A. If no such matrix B can be found, then A is said to be ***singular***.

EXAMPLE 5 An Invertible Matrix

Let

$$A = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$$

Then

$$AB = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$BA = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Thus, A and B are invertible and each is an inverse of the other.

THEOREM 1.4.4 *If B and C are both inverses of the matrix A, then $B = C$.*

Proof Since B is an inverse of A, we have $BA = I$. Multiplying both sides on the right by C gives $(BA)C = IC = C$. But it is also true that $(BA)C = B(AC) = BI = B$, so $C = B$. ◀

As a consequence of this important result, we can now speak of "the" inverse of an invertible matrix. If A is invertible, then its inverse will be denoted by the symbol A^{-1}. Thus,

$$AA^{-1} = I \quad \text{and} \quad A^{-1}A = I \tag{1}$$

THEOREM 1.4.5 *The matrix*

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is invertible if and only if $ad - bc \neq 0$, *in which case the inverse is given by the formula*

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \tag{2}$$

▶ EXAMPLE 7 Calculating the Inverse of a 2 × 2 Matrix

In each part, determine whether the matrix is invertible. If so, find its inverse.

$$\text{(a) } A = \begin{bmatrix} 6 & 1 \\ 5 & 2 \end{bmatrix} \qquad \text{(b) } A = \begin{bmatrix} -1 & 2 \\ 3 & -6 \end{bmatrix}$$

Solution (a) The determinant of A is $\det(A) = (6)(2) - (1)(5) = 7$, which is nonzero. Thus, A is invertible, and its inverse is

$$A^{-1} = \frac{1}{7}\begin{bmatrix} 2 & -1 \\ -5 & 6 \end{bmatrix} = \begin{bmatrix} \frac{2}{7} & -\frac{1}{7} \\ -\frac{5}{7} & \frac{6}{7} \end{bmatrix}$$

We leave it for you to confirm that $AA^{-1} = A^{-1}A = I$.

Solution (b) The matrix is not invertible since $\det(A) = (-1)(-6) - (2)(3) = 0$.

THEOREM 1.4.6 *If A and B are invertible matrices with the same size, then AB is invertible and*

$$(AB)^{-1} = B^{-1}A^{-1}$$

$$(AB)^{-1} = B^{-1}A^{-1}$$

$$(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$$

A product of any number of invertible matrices is invertible, and the inverse of the product is the product of the inverses in the reverse order.

Powers of a Matrix If A is a *square* matrix, then we define the nonnegative integer powers of A to be

$$A^0 = I \quad \text{and} \quad A^n = AA\cdots A \quad [n \text{ factors}]$$

and if A is invertible, then we define the negative integer powers of A to be

$$A^{-n} = (A^{-1})^n = A^{-1}A^{-1}\cdots A^{-1} \quad [n \text{ factors}]$$

Because these definitions parallel those for real numbers, the usual laws of nonnegative exponents hold; for example,

$$A^rA^s = A^{r+s} \quad \text{and} \quad (A^r)^s = A^{rs}$$

In addition, we have the following properties of negative exponents.

THEOREM 1.4.7 *If A is invertible and n is a nonnegative integer, then:*

(*a*) *A^{-1} is invertible and $(A^{-1})^{-1} = A$.*

(*b*) *A^n is invertible and $(A^n)^{-1} = A^{-n} = (A^{-1})^n$.*

(*c*) *kA is invertible for any nonzero scalar k, and $(kA)^{-1} = k^{-1}A^{-1}$.*

EXAMPLE 10 Properties of Exponents

Let A and A^{-1} be the matrices in Example 9; that is,

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \quad \text{and} \quad A^{-1} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$$

Then

$$A^{-3} = (A^{-1})^3 = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 41 & -30 \\ -15 & 11 \end{bmatrix}$$

Also,

$$A^3 = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 11 & 30 \\ 15 & 41 \end{bmatrix}$$

so, as expected from Theorem 1.4.7(b),

$$(A^3)^{-1} = \frac{1}{(11)(41) - (30)(15)} \begin{bmatrix} 41 & -30 \\ -15 & 11 \end{bmatrix} = \begin{bmatrix} 41 & -30 \\ -15 & 11 \end{bmatrix} = (A^{-1})^3$$