# 5.2 Diagonalization

**DEFINITION 1** If $A$ and $B$ are square matrices, then we say that ***$B$ is similar to $A$*** if there is an invertible matrix $P$ such that $B = P^{-1}AP$.

Note that if $B$ is similar to $A$, then it is also true that $A$ is similar to $B$ since we can express $A$ as $A = Q^{-1}BQ$ by taking $Q = P^{-1}$. This being the case, we will usually say that $A$ and $B$ are ***similar matrices*** if either is similar to the other.

In general, any property that is preserved by a similarity is called a ***similarity invariant*** and is said to be ***invariant under similarity***. Table 1 lists the most important similarity invariants. The proofs of some of these are given as exercises.

Table 1 Similarity Invariants

| Property | Description |
|---|---|
| Determinant | $A$ and $P^{-1}AP$ have the same determinant. |
| Invertibility | $A$ is invertible if and only if $P^{-1}AP$ is invertible. |
| Rank | $A$ and $P^{-1}AP$ have the same rank. |
| Nullity | $A$ and $P^{-1}AP$ have the same nullity. |
| Trace | $A$ and $P^{-1}AP$ have the same trace. |
| Characteristic polynomial | $A$ and $P^{-1}AP$ have the same characteristic polynomial. |
| Eigenvalues | $A$ and $P^{-1}AP$ have the same eigenvalues. |
| Eigenspace dimension | If $\lambda$ is an eigenvalue of $A$ (and hence of $P^{-1}AP$) then the eigenspace of $A$ corresponding to $\lambda$ and the eigenspace of $P^{-1}AP$ corresponding to $\lambda$ have the same dimension. |

**DEFINITION 2** A square matrix $A$ is said to be ***diagonalizable*** if it is similar to some diagonal matrix; that is, if there exists an invertible matrix $P$ such that $P^{-1}AP$ is diagonal. In this case the matrix $P$ is said to ***diagonalize*** $A$.

**THEOREM 5.2.1** *If $A$ is an $n \times n$ matrix, the following statements are equivalent.*

(*a*) *$A$ is diagonalizable.*

(*b*) *$A$ has $n$ linearly independent eigenvectors.*

**THEOREM 5.2.2**

(*a*) *If $\lambda_1, \lambda_2, \ldots, \lambda_k$ are distinct eigenvalues of a matrix $A$, and if $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are corresponding eigenvectors, then $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a linearly independent set.*

(*b*) *An $n \times n$ matrix with $n$ distinct eigenvalues is diagonalizable.*

**Remark** Part (*a*) of Theorem 5.2.2 is a special case of a more general result: Specifically, if $\lambda_1, \lambda_2, \ldots, \lambda_k$ are distinct eigenvalues, and if $S_1, S_2, \ldots, S_k$ are corresponding sets of linearly independent eigenvectors, then the *union* of these sets is linearly independent.

*Procedure for Diagonalizing a Matrix*

**A Procedure for Diagonalizing an $n \times n$ Matrix**

*Step 1.* Determine first whether the matrix is actually diagonalizable by searching for $n$ linearly independent eigenvectors. One way to do this is to find a basis for each eigenspace and count the total number of vectors obtained. If there is a total of $n$ vectors, then the matrix is diagonalizable, and if the total is less than $n$, then it is not.

*Step 2.* If you ascertained that the matrix is diagonalizable, then form the matrix $P = [\mathbf{p}_1 \quad \mathbf{p}_2 \quad \cdots \quad \mathbf{p}_n]$ whose column vectors are the $n$ basis vectors you obtained in Step 1.

*Step 3.* $P^{-1}AP$ will be a diagonal matrix whose successive diagonal entries are the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ that correspond to the successive columns of $P$.

**EXAMPLE 1 Finding a Matrix $P$ That Diagonalizes a Matrix $A$**

Find a matrix $P$ that diagonalizes

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

***Solution*** In Example 7 of the preceding section we found the characteristic equation of $A$ to be

$$(\lambda - 1)(\lambda - 2)^2 = 0$$

and we found the following bases for the eigenspaces:

$$\lambda = 2: \quad \mathbf{p}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{p}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}; \qquad \lambda = 1: \quad \mathbf{p}_3 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

There are three basis vectors in total, so the matrix

$$P = \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

diagonalizes $A$. As a check, you should verify that

$$P^{-1}AP = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ ◀

**Remark**

In general, there is no preferred order for the columns of $P$. Since the $i$th diagonal entry of $P^{-1}AP$ is an eigenvalue for the $i$th column vector of $P$, changing the order of the columns of $P$ just changes the order of the eigenvalues on the diagonal of $P^{-1}AP$. Thus, had we written

$$P = \begin{bmatrix} -1 & -2 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

in the preceding example, we would have obtained

$$P^{-1}AP = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

**EXAMPLE 2 A Matrix That Is Not Diagonalizable**

Show that the following matrix is not diagonalizable:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ -3 & 5 & 2 \end{bmatrix}$$

***Solution*** The characteristic polynomial of $A$ is

$$\det(\lambda I - A) = \begin{vmatrix} \lambda - 1 & 0 & 0 \\ -1 & \lambda - 2 & 0 \\ 3 & -5 & \lambda - 2 \end{vmatrix} = (\lambda - 1)(\lambda - 2)^2$$

so the characteristic equation is

$$(\lambda - 1)(\lambda - 2)^2 = 0$$

and the distinct eigenvalues of $A$ are $\lambda = 1$ and $\lambda = 2$. We leave it for you to show that bases for the eigenspaces are

$$\lambda = 1: \quad \mathbf{p}_1 = \begin{bmatrix} \frac{1}{8} \\ -\frac{1}{8} \\ 1 \end{bmatrix}; \qquad \lambda = 2: \quad \mathbf{p}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Since $A$ is a $3 \times 3$ matrix and there are only two basis vectors in total, $A$ is not diagonalizable.

**EXAMPLE 3 Recognizing Diagonalizability**

We saw in Example 3 of the preceding section that

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{bmatrix}$$

has three distinct eigenvalues: $\lambda = 4$, $\lambda = 2 + \sqrt{3}$, and $\lambda = 2 - \sqrt{3}$. Therefore, $A$ is diagonalizable and

$$P^{-1}AP = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2+\sqrt{3} & 0 \\ 0 & 0 & 2-\sqrt{3} \end{bmatrix}$$

for some invertible matrix $P$. If needed, the matrix $P$ can be found using the method shown in Example 1 of this section.

**EXAMPLE 4 Diagonalizability of Triangular Matrices**

From Theorem 5.1.2, the eigenvalues of a triangular matrix are the entries on its main diagonal. Thus, a triangular matrix with distinct entries on the main diagonal is diagonalizable. For example,

$$A = \begin{bmatrix} -1 & 2 & 4 & 0 \\ 0 & 3 & 1 & 7 \\ 0 & 0 & 5 & 8 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

is a diagonalizable matrix with eigenvalues $\lambda_1 = -1$, $\lambda_2 = 3$, $\lambda_3 = 5$, $\lambda_4 = -2$.

## *Eigenvalues of Powers of a Matrix*

**THEOREM 5.2.3** *If $k$ is a positive integer, $\lambda$ is an eigenvalue of a matrix $A$, and $\mathbf{x}$ is a corresponding eigenvector, then $\lambda^k$ is an eigenvalue of $A^k$ and $\mathbf{x}$ is a corresponding eigenvector.*

### ▶ EXAMPLE 5 Eigenvalues and Eigenvectors of Matrix Powers

In Example 2 we found the eigenvalues and corresponding eigenvectors of the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ -3 & 5 & 2 \end{bmatrix}$$

Do the same for $A^7$.

***Solution*** We know from Example 2 that the eigenvalues of $A$ are $\lambda = 1$ and $\lambda = 2$, so the eigenvalues of $A^7$ are $\lambda = 1^7 = 1$ and $\lambda = 2^7 = 128$. The eigenvectors $\mathbf{p}_1$ and $\mathbf{p}_2$ obtained in Example 1 corresponding to the eigenvalues $\lambda = 1$ and $\lambda = 2$ of $A$ are also the eigenvectors corresponding to the eigenvalues $\lambda = 1$ and $\lambda = 128$ of $A^7$. ◀

*Computing Powers of a Matrix*

The problem of computing powers of a matrix is greatly simplified when the matrix is diagonalizable. To see why this is so, suppose that $A$ is a diagonalizable $n \times n$ matrix, that $P$ diagonalizes $A$, and that

$$P^{-1}AP = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} = D$$

Squaring both sides of this equation yields

$$(P^{-1}AP)^2 = \begin{bmatrix} \lambda_1^2 & 0 & \cdots & 0 \\ 0 & \lambda_2^2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n^2 \end{bmatrix} = D^2$$

We can rewrite the left side of this equation as

$$(P^{-1}AP)^2 = P^{-1}APP^{-1}AP = P^{-1}AIAP = P^{-1}A^2P$$

from which we obtain the relationship $P^{-1}A^2P = D^2$. More generally, if $k$ is a positive integer, then a similar computation will show that

$$P^{-1}A^kP = D^k = \begin{bmatrix} \lambda_1^k & 0 & \cdots & 0 \\ 0 & \lambda_2^k & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n^k \end{bmatrix}$$

which we can rewrite as

$$A^k = PD^kP^{-1} = P \begin{bmatrix} \lambda_1^k & 0 & \cdots & 0 \\ 0 & \lambda_2^k & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n^k \end{bmatrix} P^{-1} \tag{3}$$

Formula (3) reveals that raising a diagonalizable matrix $A$ to a positive integer power has the effect of raising its eigenvalues to that power.

**EXAMPLE 6 Powers of a Matrix**

Use (3) to find $A^{13}$, where

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

***Solution*** We showed in Example 1 that the matrix $A$ is diagonalized by

$$P = \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

and that

$$D = P^{-1}AP = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus, it follows from (3) that

$$A^{13} = PD^{13}P^{-1} = \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2^{13} & 0 & 0 \\ 0 & 2^{13} & 0 \\ 0 & 0 & 1^{13} \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ -1 & 0 & -1 \end{bmatrix} \tag{4}$$

$$= \begin{bmatrix} -8190 & 0 & -16382 \\ 8191 & 8192 & 8191 \\ 8191 & 0 & 16383 \end{bmatrix}$$

**Remark** With the method in the preceding example, most of the work is in diagonalizing $A$. Once that work is done, it can be used to compute any power of $A$. Thus, to compute $A^{1000}$ we need only change the exponents from 13 to 1000 in (4).

**Definition of geometric multiplicity and algebraic multiplicity:** If $\lambda_0$ is an eigenvalue of an $n \times n$ matrix $A$, then the dimension of the eigenspace corresponding to $\lambda_0$ is called the ***geometric multiplicity*** of $\lambda_0$, and the number of times that $\lambda - \lambda_0$ appears as a factor in the characteristic polynomial of $A$ is called the ***algebraic multiplicity*** of $\lambda_0$.

**Remark**

If $\lambda$ is an eigenvalue of the matrix $A \in M_n(\mathbb{R})$, then $E_\lambda = \{X \in \mathbb{R}^n;\ AX = \lambda X\}$ is vector sub-space of $\mathbb{R}^n$. Its dimension is called the the geometric multiplicity of $\lambda$.

**Definition**

If $A \in M_n(\mathbb{R})$ and the characteristic function

$$\det(\lambda I - A) = (\lambda - \lambda_1)^m Q(\lambda)$$

such that $Q(\lambda_1) \neq 0$ we say that $m$ is the algebraic multiplicity of the eigenvalue $\lambda_1$.

**THEOREM 5.2.4 Geometric and Algebraic Multiplicity**

*If $A$ is a square matrix, then:*

(a) *For every eigenvalue of $A$, the geometric multiplicity is less than or equal to the algebraic multiplicity.*

(b) *$A$ is diagonalizable if and only if the geometric multiplicity of every eigenvalue is equal to the algebraic multiplicity.*

**Remark**

Special case

If $A \in M_n(\mathbb{R})$ and has $n$ different eigenvalues, then $A$ is diagonalizable.

**Exercise**

Show if the following matrix is diagonalizable and find the matrix $P$ such that the matrix $P^{-1}AP$ is diagonal.

$$A = \begin{pmatrix} -10 & -6 \\ 18 & 11 \end{pmatrix}$$

**Solution** The characteristic function of the matrix $A$ is

$$\det(\lambda I - A) = \begin{vmatrix} -10-\lambda & -6 \\ 18 & 11-\lambda \end{vmatrix} = (\lambda - 2)(1 + \lambda).$$

Then the matrix is diagonalizable.

$E_{-1} = \langle(-2, 3)\rangle$ and $E_2 = \langle(1, -2)\rangle$.

The diagonal matrix is $D = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}$

and the matrix $P$ is $P = \begin{pmatrix} -2 & 1 \\ 3 & -2 \end{pmatrix}$.

**Exercise**

Show if the following matrix is diagonalizable and find the matrix $P$ such that the matrix $P^{-1}AP$ is diagonal.

$$A = \begin{pmatrix} 5 & -3 & 0 & 9 \\ 0 & 3 & 1 & -2 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

**Solution** The characteristic function of the matrix $A$ is

$$\det(\lambda I - A) = \begin{vmatrix} 5-\lambda & -3 & 0 & 9 \\ 0 & 3-\lambda & 1 & -2 \\ 0 & 0 & 2-\lambda & 0 \\ 0 & 0 & 0 & 2-\lambda \end{vmatrix} = (5-\lambda)(3-\lambda)(2-\lambda)^2.$$

The matrix is diagonalizable if and only if the dimension of the vector space $E_2$ is 2.

$E_2 = \langle (1, 1, -1, 0), (-1, 2, 0, 1) \rangle$.

Then the matrix $A$ is diagonalizable.

$E_5 = \langle (1, 0, 0, 0) \rangle$ and $E_3 = \langle (3, 2, 0, 0) \rangle$.

The diagonal matrix is $D = \begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$

and the matrix $P$ is $P = \begin{pmatrix} 1 & 3 & 1 & -1 \\ 0 & 2 & 1 & 2 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

**Exercise**

Show if the following matrix is diagonalizable and find the matrix $P$ such that the matrix $P^{-1}AP$ is diagonal.

$$A = \begin{pmatrix} 2 & -1 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} \\ -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 3 \end{pmatrix}$$

**Solution** The characteristic function of the matrix $A$ is

$$\det(\lambda I - A) = \begin{vmatrix} 2-\lambda & -1 & 0 & \frac{1}{2} \\ 0 & 1-\lambda & 0 & \frac{1}{2} \\ -1 & 1 & 1-\lambda & -1 \\ 1 & -1 & 1 & 3-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda)^3.$$

The matrix is diagonalizable if and only if the dimension the vector space $E_2$ is 3.

$E_2 = \langle(-1, 1, 0, 2), (-1, 0, 1, 0)\rangle$.

Thus, the matrix A is not diagonalizable.