Kernel and Range

DEFINITION 2 If $T: V \to W$ is a linear transformation, then the set of vectors in V that T maps into $\mathbf{0}$ is called the ***kernel*** of T and is denoted by $\ker(T)$. The set of all vectors in W that are images under T of at least one vector in V is called the ***range*** of T and is denoted by $R(T)$, or it is called the image of T, $\mathrm{Im}(T)$.

EXAMPLE 13 Kernel and Range of a Matrix Transformation

If $A \in M_{m,n}(\mathbb{R})$ and $T_A\colon \mathbb{R}^n \longrightarrow \mathbb{R}^m$ the linear transformation defined by: $T_A(X) = AX$, then $\mathrm{rank}(T) = \mathrm{rank}A$, and $\mathrm{Im}(T) = \mathrm{col}A$.

EXAMPLE 14 Kernel and Range of the Zero Transformation

Let $T: V \to W$ be the zero transformation. Since T maps *every* vector in V into $\mathbf{0}$, it follows that $\ker(T) = V$. Moreover, since $\mathbf{0}$ is the *only* image under T of vectors in V, it follows that $R(T) = \{\mathbf{0}\}$.

EXAMPLE 15 Kernel and Range of the Identity Operator

Let $I: V \to V$ be the identity operator. Since $I(\mathbf{v}) = \mathbf{v}$ for all vectors in V, *every* vector in V is the image of some vector (namely, itself); thus $R(I) = V$. Since the *only* vector that I maps into $\mathbf{0}$ is $\mathbf{0}$, it follows that $\ker(I) = \{\mathbf{0}\}$.

Properties of Kernel and Range

THEOREM 8.1.3 *If $T: V \to W$ is a linear transformation, then:*

(*a*) *The kernel of T is a subspace of V.*

(*b*) *The range of T is a subspace of W.*

Proof (a) To show that $\ker(T)$ is a subspace, we must show that it contains at least one vector and is closed under addition and scalar multiplication. By part (*a*) of Theorem 8.1.1, the vector $\mathbf{0}$ is in $\ker(T)$, so the kernel contains at least one vector. Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be vectors in $\ker(T)$, and let k be any scalar. Then

$$T(\mathbf{v}_1 + \mathbf{v}_2) = T(\mathbf{v}_1) + T(\mathbf{v}_2) = \mathbf{0} + \mathbf{0} = \mathbf{0}$$

so $\mathbf{v}_1 + \mathbf{v}_2$ is in $\ker(T)$. Also,

$$T(k\mathbf{v}_1) = kT(\mathbf{v}_1) = k\mathbf{0} = \mathbf{0}$$

so $k\mathbf{v}_1$ is in $\ker(T)$.

Proof (b) To show that $R(T)$ is a subspace of W, we must show that it contains at least one vector and is closed under addition and scalar multiplication. However, it contains at least the zero vector of W since $T(\mathbf{0}) = (\mathbf{0})$ by part (*a*) of Theorem 8.1.1. To prove that it is closed under addition and scalar multiplication, we must show that if $\mathbf{w}_1$ and $\mathbf{w}_2$ are vectors in $R(T)$, and if k is any scalar, then there exist vectors $\mathbf{a}$ and $\mathbf{b}$ in V for which

$$T(\mathbf{a}) = \mathbf{w}_1 + \mathbf{w}_2 \quad \text{and} \quad T(\mathbf{b}) = k\mathbf{w}_1 \tag{4}$$

But the fact that $\mathbf{w}_1$ and $\mathbf{w}_2$ are in $R(T)$ tells us there exist vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ in V such that

$$T(\mathbf{v}_1) = \mathbf{w}_1 \quad \text{and} \quad T(\mathbf{v}_2) = \mathbf{w}_2$$

The following computations complete the proof by showing that the vectors $\mathbf{a} = \mathbf{v}_1 + \mathbf{v}_2$ and $\mathbf{b} = k\mathbf{v}_1$ satisfy the equations in (4):

$$T(\mathbf{a}) = T(\mathbf{v}_1 + \mathbf{v}_2) = T(\mathbf{v}_1) + T(\mathbf{v}_2) = \mathbf{w}_1 + \mathbf{w}_2$$
$$T(\mathbf{b}) = T(k\mathbf{v}_1) = kT(\mathbf{v}_1) = k\mathbf{w}_1 \quad ◀$$

Rank and Nullity of Linear Transformations

DEFINITION 3 Let $T: V \to W$ be a linear transformation. If the range of T is finite-dimensional, then its dimension is called the ***rank of T***; and if the kernel of T is finite-dimensional, then its dimension is called the ***nullity of T***. The rank of T is denoted by $\text{rank}(T)$ and the nullity of T by $\text{nullity}(T)$.

THEOREM 8.1.4 Dimension Theorem for Linear Transformations

If $T: V \to W$ is a linear transformation from a finite-dimensional vector space V to a vector space W, then the range of T is finite-dimensional, and

$$\mathit{rank}(T) + \mathit{nullity}(T) = \dim(V) \tag{7}$$

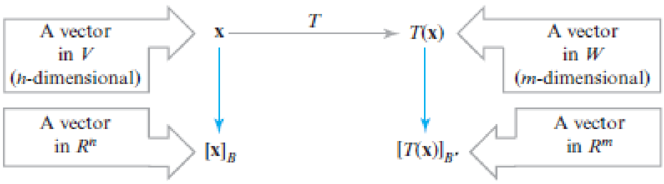
In the special case where A is an $m \times n$ matrix and $T_A: R^n \to R^m$ is multiplication by A, the kernel of T_A is the null space of A, and the range of T_A is the column space of A. Thus, it follows from Theorem 8.1.4 that

$$\text{rank}(T_A) + \text{nullity}(T_A) = n$$

8.4 Matrices for General Linear Transformations

Matrices of Linear Transformations

Suppose that V is an n-dimensional vector space, that W is an m-dimensional vector space, and that $T: V \to W$ is a linear transformation. Suppose further that B is a basis for V, that B' is a basis for W, and that for each vector $\mathbf{x}$ in V, the coordinate matrices for $\mathbf{x}$ and $T(\mathbf{x})$ are $[\mathbf{x}]_B$ and $[T(\mathbf{x})]_{B'}$, respectively (Figure 8.4.1).



▶ Figure 8.4.1

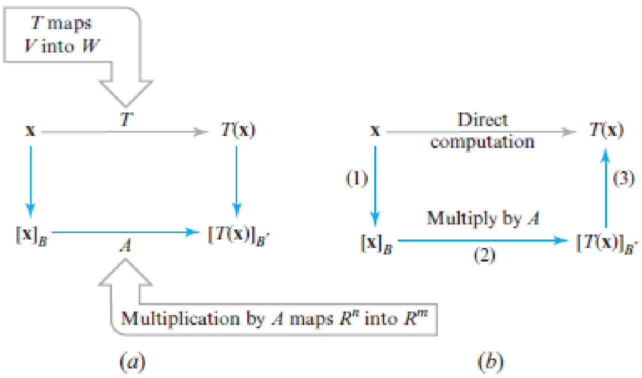
It will be our goal to find an $m \times n$ matrix A such that multiplication by A maps the vector $[\mathbf{x}]_B$ into the vector $[T(\mathbf{x})]_{B'}$ for each $\mathbf{x}$ in V (Figure 8.4.2a). If we can do so, then, as illustrated in Figure 8.4.2b, we will be able to execute the linear transformation T by using matrix multiplication and the following *indirect* procedure:

Finding $T(\mathbf{x})$ Indirectly

Step 1. Compute the coordinate vector $[\mathbf{x}]_B$.

Step 2. Multiply $[\mathbf{x}]_B$ on the left by A to produce $[T(\mathbf{x})]_{B'}$.

Step 3. Reconstruct $T(\mathbf{x})$ from its coordinate vector $[T(\mathbf{x})]_{B'}$.



▶ Figure 8.4.2

Theorem

Let $T\colon V \longrightarrow W$ be a linear transformation and let $B = (u_1, \dots, u_n)$ be a basis of the vector space V and $C = (v_1, \dots, v_m)$ basis of the vector space W. Then there is a unique matrix $[T]_B^C$ such that its columns $[T(u_1)]_C, \dots, [T(u_n)]_C$. The matrix $[T]_B^C$ is called the matrix of the linear transformation T with respect to the basis B and the basis C. and satisfies

$$[T(v)]_C = [T]_B^C \, [v]_B\,; \qquad \forall v \in V.$$

If $V = W$ and $B = C$ we write the matrix $[T]_C$ instead of $[T]_B^C$.

Remark. In the book, the matrix $[T]_B^C$ is denoted by $[T]_{C,B}$

that is, $[T]_{C,B} = [T]_B^C$

EXAMPLE 1 Matrix for a Linear Transformation

Let $T: P_1 \rightarrow P_2$ be the linear transformation defined by

$$T(p(x)) = xp(x)$$

Find the matrix for T with respect to the standard bases

$$B = \{\mathbf{u}_1, \mathbf{u}_2\} \quad \text{and} \quad B' = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$$

where

$$\mathbf{u}_1 = 1, \quad \mathbf{u}_2 = x; \qquad \mathbf{v}_1 = 1, \quad \mathbf{v}_2 = x, \quad \mathbf{v}_3 = x^2$$

Solution From the given formula for T we obtain

$$T(\mathbf{u}_1) = T(1) = (x)(1) = x$$
$$T(\mathbf{u}_2) = T(x) = (x)(x) = x^2$$

By inspection, the coordinate vectors for $T(\mathbf{u}_1)$ and $T(\mathbf{u}_2)$ relative to B' are

$$[T(\mathbf{u}_1)]_{B'} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad [T(\mathbf{u}_2)]_{B'} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

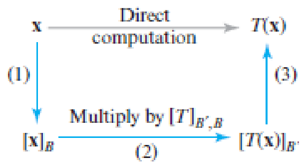
Thus, the matrix for T with respect to B and B' is

$$[T]_B^{B'} = [T]_{B',B} = \left[[T(\mathbf{u}_1)]_{B'} \mid [T(\mathbf{u}_2)]_{B'} \right] = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

▶ EXAMPLE 2 The Three-Step Procedure

Let $T: P_1 \rightarrow P_2$ be the linear transformation in Example 1, and use the three-step procedure described in the following figure to perform the computation

$$T(a + bx) = x(a + bx) = ax + bx^2$$



Solution

Step 1. The coordinate matrix for $\mathbf{x} = a + bx$ relative to the basis $B = \{1, x\}$ is

$$[\mathbf{x}]_B = \begin{bmatrix} a \\ b \end{bmatrix}$$

Step 2. Multiplying $[\mathbf{x}]_B$ by the matrix $[T]_{B',B}$ found in Example 1 we obtain

$$[T]_{B',B}[\mathbf{x}]_B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ a \\ b \end{bmatrix} = [T(\mathbf{x})]_{B'}$$

Step 3. Reconstructing $T(\mathbf{x}) = T(a + bx)$ from $[T(\mathbf{x})]_{B'}$ we obtain

$$T(a + bx) = 0 + ax + bx^2 = ax + bx^2$$

▶ EXAMPLE 3 Matrix for a Linear Transformation

Let $T : R^2 \to R^3$ be the linear transformation defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_2 \\ -5x_1 + 13x_2 \\ -7x_1 + 16x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -5 & 13 \\ -7 & 16 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Find the matrix for the transformation T with respect to the bases $B = \{\mathbf{u}_1, \mathbf{u}_2\}$ for R^2 and $B' = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ for R^3, where

$$\mathbf{u}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 5 \\ 2 \end{bmatrix}; \qquad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

Solution From the formula for T,

$$T(\mathbf{u}_1) = \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}, \quad T(\mathbf{u}_2) = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}$$

Expressing these vectors as linear combinations of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$, we obtain (verify)

$$T(\mathbf{u}_1) = \mathbf{v}_1 - 2\mathbf{v}_3, \quad T(\mathbf{u}_2) = 3\mathbf{v}_1 + \mathbf{v}_2 - \mathbf{v}_3$$

Thus,

$$[T(\mathbf{u}_1)]_{B'} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \quad [T(\mathbf{u}_2)]_{B'} = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$$

so

$$[T]_B^{B'} = [T]_{B',B} = \left[[T(\mathbf{u}_1)]_{B'} \mid [T(\mathbf{u}_2)]_{B'}\right] = \begin{bmatrix} 1 & 3 \\ 0 & 1 \\ -2 & -1 \end{bmatrix}$$ ◀

Remark Example 3 illustrates that a fixed linear transformation generally has multiple representations, each depending on the bases chosen. In this case the matrices

$$[T] = \begin{bmatrix} 0 & 1 \\ -5 & 13 \\ -7 & 16 \end{bmatrix} \quad \text{and} \quad [T]_{B',B} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \\ -2 & -1 \end{bmatrix}$$

both represent the transformation T, the first relative to the standard bases for R^2 and R^3, the second relative to the bases B and B' stated in the example.

Matrices of Linear Operators

In the special case where $V = W$ (so that $T : V \to V$ is a linear operator), it is usual to take $B = B'$ when constructing a matrix for T. In this case the resulting matrix is called the ***matrix for T relative to the basis B*** and is usually denoted by $[T]_B$ rather than $[T]_{B,B}$. If $B = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$, then Formulas (4) and (5) become

$$[T]_B = \big[[T(\mathbf{u}_1)]_B \mid [T(\mathbf{u}_2)]_B \mid \cdots \mid [T(\mathbf{u}_n)]_B\big] \tag{7}$$

$$[T]_B[\mathbf{x}]_B = [T(\mathbf{x})]_B \tag{8}$$

In the special case where $T: R^n \to R^n$ is a matrix operator, say multiplication by A, and B is the standard basis for R^n, then Formula (7) simplifies to

$$[T]_B = A \tag{9}$$

Recall that the identity operator $I: V \to V$ maps every vector in V into itself, that is, $I(\mathbf{x}) = \mathbf{x}$ for every vector $\mathbf{x}$ in V. The following example shows that if V is n-dimensional, then the matrix for I relative to *any* basis B for V is the $n \times n$ identity matrix.

▶ EXAMPLE 4 Matrices of Identity Operators

If $B = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ is a basis for a finite-dimensional vector space V, and if $I: V \to V$ is the identity operator on V, then

$$I(\mathbf{u}_1) = \mathbf{u}_1, \quad I(\mathbf{u}_2) = \mathbf{u}_2, \ldots, \quad I(\mathbf{u}_n) = \mathbf{u}_n$$

Therefore,

$$[I]_B = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} = I$$

$$\uparrow \qquad \uparrow \qquad \uparrow$$

$$[I(\mathbf{u}_1)]_B \quad [I(\mathbf{u}_2)]_B \quad [I(\mathbf{u}_n)]_B$$

EXAMPLE 5 Linear Operator on P_2

Let $T: P_2 \to P_2$ be the linear operator defined by

$$T(p(x)) = p(3x-5)$$

that is, $T(c_0 + c_1x + c_2x^2) = c_0 + c_1(3x-5) + c_2(3x-5)^2$.

(a) Find $[T]_B$ relative to the basis $B = \{1, x, x^2\}$.

(b) Use the indirect procedure to compute $T(1+2x+3x^2)$.

(c) Check the result in (b) by computing $T(1+2x+3x^2)$ directly.

Solution (a) From the formula for T,

$$T(1) = 1, \quad T(x) = 3x-5, \quad T(x^2) = (3x-5)^2 = 9x^2 - 30x + 25$$

so

$$[T(1)]_B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad [T(x)]_B = \begin{bmatrix} -5 \\ 3 \\ 0 \end{bmatrix}, \quad [T(x^2)]_B = \begin{bmatrix} 25 \\ -30 \\ 9 \end{bmatrix}$$

Thus,

$$[T]_B = \begin{bmatrix} 1 & -5 & 25 \\ 0 & 3 & -30 \\ 0 & 0 & 9 \end{bmatrix}$$

Solution (b)

Step 1. The coordinate matrix for $\mathbf{p} = 1 + 2x + 3x^2$ relative to the basis $B = \{1, x, x^2\}$ is

$$[\mathbf{p}]_B = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Step 2. Multiplying $[\mathbf{p}]_B$ by the matrix $[T]_B$ found in part (a) we obtain

$$[T]_B[\mathbf{p}]_B = \begin{bmatrix} 1 & -5 & 25 \\ 0 & 3 & -30 \\ 0 & 0 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 66 \\ -84 \\ 27 \end{bmatrix} = [T(\mathbf{p})]_B$$

Step 3. Reconstructing $T(\mathbf{p}) = T(1+2x+3x^2)$ from $[T(\mathbf{p})]_B$ we obtain

$$T(1+2x+3x^2) = 66 - 84x + 27x^2$$

Solution (c) By direct computation,

$$\begin{aligned} T(1+2x+3x^2) &= 1 + 2(3x-5) + 3(3x-5)^2 \\ &= 1 + 6x - 10 + 27x^2 - 90x + 75 \\ &= 66 - 84x + 27x^2 \end{aligned}$$

which agrees with the result in (b). ◀

Theorem

If $T: V \longrightarrow V$ is a linear transformation and B and C are basis of the vector space V, then

$$[T]_B = {}_BP_C\,[T]_C\,{}_CP_B.$$

where ${}_BP_C$ is the transition matrix from C to B, and ${}_CP_B$ is the transition matrix from B to C.

and remember that ${}_CP_B^{-1} = {}_BP_C$.

▶ **EXAMPLE**

Let $T\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be the linear transformation such that its matrix with respect to the standard basis C of the vector space $\mathbb{R}^3$ is

$$[T]_C = \begin{pmatrix} -3 & 2 & 2 \\ -5 & 4 & 2 \\ 1 & -1 & 1 \end{pmatrix}$$

Find the matrix of the linear transformation $[T]_B$ with respect to the following basis B

$$B = \{u = (1,1,1),\ v = (1,1,0),\ w = (0,1,-1)\}.$$

We need to find the transition matrix

$$_C P_B = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix}$$

$$_B P_C = {}_C P_B^{-1} = \begin{pmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ -1 & 1 & 0 \end{pmatrix}$$

and

$$[T]_B = {}_B P_C [T]_C\, {}_C P_B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Question

Let B and C be bases of a vector space V of dimension 3 such that ${}_CP_B = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$.
(${}_CP_B$ is the transition matrix from the basis B to the basis C). Let $T\colon V \longrightarrow V$ be a linear transformation with $[T]_B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 1 & 0 \end{pmatrix}$.

1. If $[v]_C = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, find $[v]_B$.
2. If $[w]_B = \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}$, find $[T(w)]_B$.
3. If $B = \{u_1, u_2, u_3\}$. Find the values of a, b, c such that $T(u_1) = au_1 - \frac{b}{5}u_2 + cu_3$.

Solution

1. ${}_BP_C = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$. $[v]_B = {}_BP_C[v]_C = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$
2. $[T(w)]_B = [T]_B[w]_B = \begin{pmatrix} 1 \\ -3 \\ -2 \end{pmatrix}$.
3. $T(u_1) = u_1 + u_2 + 2u_3 = au_1 - \frac{b}{5}u_2 + cu_3$. Then $a = 1$, $b = -5$, $c = 2$.

Question

Define $T\colon \mathbb{R}^4 \longrightarrow \mathbb{R}^3$ by:

$$T(x, y, z, t) = (x - 2y + z + 3t, 2x - 3y + 2t, -x + 3z + 5t).$$

(a) Find the matrix of the linear transformation T with respect to the standard bases of $\mathbb{R}^4$ and $\mathbb{R}^3$.

(b) Find a basis for kernel T.

(c) Find a basis for Image T.

Solution

2. (a) The standard matrix of T is $\begin{pmatrix} 1 & -2 & 1 & 3 \\ 2 & -3 & 0 & 2 \\ -1 & 0 & 3 & 5 \end{pmatrix}$.

(b) The reduced echelon form of to the matrix of T is $\begin{pmatrix} 1 & 0 & -3 & -5 \\ 0 & 1 & -2 & -4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. Then $\{(3, 2, 1, 0), (5, 4, 0, 1)\}$ is a basis for kernel T.

(c) Using the reduced echelon form of to the matrix of T we deduce that $\{(1, 2, -1), (2, 3, 0)\}$ is a basis for Image T.