Coordinate System and Change of Bases

Definition

If $S = \{v_1, \ldots, v_n\}$ is a basis of the vector space V and if $v \in V$ such that

$$v = x_1 v_1 + \ldots x_n v_n$$

then $(x_1, \ldots x_n)$ are called the system of coordinates of the vector v in the basis S. We denote $[v]_S = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

Theorem

If $B = \{v_1, \dots, v_n\}$ and $C = \{u_1, \dots, u_n\}$ are two bases of the vector space V. The matrix ${}_C P_B \in M_n(\mathbb{R})$ with columns $[v_1]_C, \dots, [v_n]_C$ is called the change of bases matrix from the basis B to the basis C. This matrix ${}_C P_B$ is invertible, ${}_C P_B^{-1} = {}_B P_C$ and

$$[v]_C = {}_C P_B [v]_B, \quad \text{for all } v \in V.$$

(${}_B P_C$ is the change of bases matrix from the basis C to the basis B.)

Note: The notation used in the book for the above definition is:

Transition Matrices The matrix P in Equation (7) is called the ***transition matrix*** from B' to B. For emphasis, we will often denote it by $P_{B' \to B}$. It follows from (8) that this matrix can be expressed in terms of its column vectors as

$$P_{B' \to B} = \left[[\mathbf{u}'_1]_B \mid [\mathbf{u}'_2]_B \mid \cdots \mid [\mathbf{u}'_n]_B \right] \tag{9}$$

Similarly, the transition matrix from B to B' can be expressed in terms of its column vectors as

$$P_{B \to B'} = \left[[\mathbf{u}_1]_{B'} \mid [\mathbf{u}_2]_{B'} \mid \cdots \mid [\mathbf{u}_n]_{B'} \right] \tag{10}$$

Remark There is a simple way to remember both of these formulas using the terms "old basis" and "new basis" defined earlier in this section: In Formula (9) the old basis is B' and the new basis is B, whereas in Formula (10) the old basis is B and the new basis is B'. Thus, both formulas can be restated as follows:

The columns of the transition matrix from an old basis to a new basis are the coordinate vectors of the old basis relative to the new basis.

An Efficient Method for Computing Transition Matrices for R^n

Our next objective is to develop an efficient procedure for computing transition matrices *between bases for* R^n. As illustrated in Example 1, the first step in computing a transition matrix is to express each new basis vector as a linear combination of the old basis vectors. For R^n this involves solving n linear systems of n equations in n unknowns, each of which has the same coefficient matrix (why?). An efficient way to do this is by the method illustrated in Example 2 of Section 1.6, which is as follows:

A Procedure for Computing $P_{B \to B'}$

Step 1. Form the matrix $[B' \mid B]$.

Step 2. Use elementary row operations to reduce the matrix in Step 1 to reduced row echelon form.

Step 3. The resulting matrix will be $[I \mid P_{B \to B'}]$.

Step 4. Extract the matrix $P_{B \to B'}$ from the right side of the matrix in Step 3.

This procedure is captured in the following diagram.

$$[\text{new basis} \mid \text{old basis}] \xrightarrow{\text{row operations}} [I \mid \text{transition from old to new}] \tag{14}$$

Example

Let $B = \{v_1 = (0, 1, 1), v_2 = (1, 0, -2), v_3 = (1, 1, 0)\}$ be a basis of the vector space $\mathbb{R}^3$ and let $C = \{u_1 = (1, 0, 0), u_2 = (0, 1, 0), u_3 = (0, 0, 1)\}$ be the standard basis of $\mathbb{R}^3$.

We have ${}_CP_B = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & -2 & 0 \end{pmatrix}$ and ${}_BP_C = \begin{pmatrix} -2 & 2 & -1 \\ -1 & 1 & -1 \\ 2 & -1 & 1 \end{pmatrix}$. If $[v]_C = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, then $[v]_B = {}_BP_C[v]_C = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$.

Consider B and C two bases of vector space V such that the matrix $_C P_B = \begin{pmatrix} 1 & -1 & -2 \\ 0 & -1 & 1 \\ 1 & 2 & -2 \end{pmatrix}$. Let u be a vector in V such that $[u]_C = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$,

then $[u]_B = {_B P_C}[u]_C = {_C P_B^{-1}}[u]_C = \frac{1}{3}\begin{pmatrix} 6y + 3z \\ -x + z \\ -x + 3y + z \end{pmatrix}$.