

## 1.3 Matrices and Matrix Operations

**DEFINITION 1** A *matrix* is a rectangular array of numbers. The numbers in the array are called the *entries* in the matrix.

### ▶ EXAMPLE 1 Examples of Matrices

Some examples of matrices are

$$\begin{bmatrix} 1 & 2 \\ 3 & 0 \\ -1 & 4 \end{bmatrix}, [2 \quad 1 \quad 0 \quad -3], \begin{bmatrix} e & \pi & -\sqrt{2} \\ 0 & \frac{1}{2} & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix}, [4] \quad \blacktriangleleft$$

The *size* of a matrix is described in terms of the number of rows (horizontal lines) and columns (vertical lines) it contains. For example, the first matrix in Example 1 has three rows and two columns, so its size is 3 by 2 (written  $3 \times 2$ ). In a size description, the first number always denotes the number of rows, and the second denotes the number of columns. The remaining matrices in Example 1 have sizes  $1 \times 4$ ,  $3 \times 3$ ,  $2 \times 1$ , and  $1 \times 1$ , respectively.

A matrix with only one row, such as the second in Example 1, is called a *row vector* (or a *row matrix*), and a matrix with only one column, such as the fourth in that example, is called a *column vector* (or a *column matrix*). The fifth matrix in that example is both a row vector and a column vector.

We will use capital letters to denote matrices and lowercase letters to denote numerical quantities; thus we might write

$$A = \begin{bmatrix} 2 & 1 & 7 \\ 3 & 4 & 2 \end{bmatrix} \quad \text{or} \quad C = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$$

The entry that occurs in row  $i$  and column  $j$  of a matrix  $A$  will be denoted by  $a_{ij}$ . Thus a general  $3 \times 4$  matrix might be written as

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

and a general  $m \times n$  matrix as

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad (1)$$

A matrix with  $n$  rows and  $n$  columns is said to be a *square matrix of order  $n$* .

The entry in row  $i$  and column  $j$  of a matrix  $A$  is also commonly denoted by the symbol  $(A)_{ij}$ . Thus, for matrix (1) above, we have

$$(A)_{ij} = a_{ij}$$

and for the matrix

$$A = \begin{bmatrix} 2 & -3 \\ 7 & 0 \end{bmatrix}$$

we have  $(A)_{11} = 2$ ,  $(A)_{12} = -3$ ,  $(A)_{21} = 7$ , and  $(A)_{22} = 0$ .

Row and column vectors are of special importance, and it is common practice to denote them by boldface lowercase letters rather than capital letters. For such matrices, double subscripting of the entries is unnecessary. Thus a general  $1 \times n$  row vector  $\mathbf{a}$  and a general  $m \times 1$  column vector  $\mathbf{b}$  would be written as

$$\mathbf{a} = [a_1 \ a_2 \ \cdots \ a_n] \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

A matrix  $A$  with  $n$  rows and  $n$  columns is called a *square matrix of order  $n$* , and the shaded entries  $a_{11}, a_{22}, \dots, a_{nn}$  in (2) are said to be on the *main diagonal* of  $A$ .

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \quad (2)$$

### Operations on Matrices

**DEFINITION 2** Two matrices are defined to be *equal* if they have the same size and their corresponding entries are equal.

► **EXAMPLE 2 Equality of Matrices**

Consider the matrices

$$A = \begin{bmatrix} 2 & 1 \\ 3 & x \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 4 & 0 \end{bmatrix}$$

If  $x = 5$ , then  $A = B$ , but for all other values of  $x$  the matrices  $A$  and  $B$  are not equal, since not all of their corresponding entries are equal. There is no value of  $x$  for which  $A = C$  since  $A$  and  $C$  have different sizes. ◀

**DEFINITION 3** If  $A$  and  $B$  are matrices of the same size, then the *sum*  $A + B$  is the matrix obtained by adding the entries of  $B$  to the corresponding entries of  $A$ , and the *difference*  $A - B$  is the matrix obtained by subtracting the entries of  $B$  from the corresponding entries of  $A$ . Matrices of different sizes cannot be added or subtracted.

► **EXAMPLE 3 Addition and Subtraction**

Consider the matrices

$$A = \begin{bmatrix} 2 & 1 & 0 & 3 \\ -1 & 0 & 2 & 4 \\ 4 & -2 & 7 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} -4 & 3 & 5 & 1 \\ 2 & 2 & 0 & -1 \\ 3 & 2 & -4 & 5 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$$

Then

$$A + B = \begin{bmatrix} -2 & 4 & 5 & 4 \\ 1 & 2 & 2 & 3 \\ 7 & 0 & 3 & 5 \end{bmatrix} \quad \text{and} \quad A - B = \begin{bmatrix} 6 & -2 & -5 & 2 \\ -3 & -2 & 2 & 5 \\ 1 & -4 & 11 & -5 \end{bmatrix}$$

The expressions  $A + C$ ,  $B + C$ ,  $A - C$ , and  $B - C$  are undefined. ◀

**DEFINITION 4** If  $A$  is any matrix and  $c$  is any scalar, then the *product*  $cA$  is the matrix obtained by multiplying each entry of the matrix  $A$  by  $c$ . The matrix  $cA$  is said to be a *scalar multiple* of  $A$ .

In matrix notation, if  $A = [a_{ij}]$ , then

$$(cA)_{ij} = c(A)_{ij} = ca_{ij}$$

► **EXAMPLE 4 Scalar Multiples**

For the matrices

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 3 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 2 & 7 \\ -1 & 3 & -5 \end{bmatrix}, \quad C = \begin{bmatrix} 9 & -6 & 3 \\ 3 & 0 & 12 \end{bmatrix}$$

we have

$$2A = \begin{bmatrix} 4 & 6 & 8 \\ 2 & 6 & 2 \end{bmatrix}, \quad (-1)B = \begin{bmatrix} 0 & -2 & -7 \\ 1 & -3 & 5 \end{bmatrix}, \quad \frac{1}{3}C = \begin{bmatrix} 3 & -2 & 1 \\ 1 & 0 & 4 \end{bmatrix}$$

It is common practice to denote  $(-1)B$  by  $-B$ . ◀

**DEFINITION 5** If  $A$  is an  $m \times r$  matrix and  $B$  is an  $r \times n$  matrix, then the *product*  $AB$  is the  $m \times n$  matrix whose entries are determined as follows: To find the entry in row  $i$  and column  $j$  of  $AB$ , single out row  $i$  from the matrix  $A$  and column  $j$  from the matrix  $B$ . Multiply the corresponding entries from the row and column together, and then add up the resulting products.

► **EXAMPLE 5 Multiplying Matrices**

Consider the matrices

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix}$$

Since  $A$  is a  $2 \times 3$  matrix and  $B$  is a  $3 \times 4$  matrix, the product  $AB$  is a  $2 \times 4$  matrix. To determine, for example, the entry in row 2 and column 3 of  $AB$ , we single out row 2 from  $A$  and column 3 from  $B$ . Then, as illustrated below, we multiply corresponding entries together and add up these products.

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix} = \begin{bmatrix} \square & \square & \square & \square \\ \square & \square & 26 & \square \end{bmatrix}$$

$$(2 \cdot 4) + (6 \cdot 3) + (0 \cdot 5) = 26$$

The entry in row 1 and column 4 of  $AB$  is computed as follows:

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix} = \begin{bmatrix} \square & \square & \square & 13 \\ \square & \square & \square & \square \end{bmatrix}$$

$$(1 \cdot 3) + (2 \cdot 1) + (4 \cdot 2) = 13$$

The computations for the remaining entries are

$$\begin{aligned} (1 \cdot 4) + (2 \cdot 0) + (4 \cdot 2) &= 12 \\ (1 \cdot 1) + (2 \cdot -1) + (4 \cdot 7) &= 27 \\ (1 \cdot 4) + (2 \cdot 3) + (4 \cdot 5) &= 30 \\ (2 \cdot 4) + (6 \cdot 0) + (0 \cdot 2) &= 8 \\ (2 \cdot 1) + (6 \cdot 1) + (0 \cdot 7) &= -4 \\ (2 \cdot 3) + (6 \cdot 1) + (0 \cdot 2) &= 12 \end{aligned} \quad AB = \begin{bmatrix} 12 & 27 & 30 & 13 \\ 8 & -4 & 26 & 12 \end{bmatrix} \blacktriangleleft$$

The definition of matrix multiplication requires that the number of columns of the first factor  $A$  be the same as the number of rows of the second factor  $B$  in order to form the product  $AB$ . If this condition is not satisfied, the product is undefined. A convenient way to determine whether a product of two matrices is defined is to write down the size of the first factor and, to the right of it, write down the size of the second factor. If, as in (3), the inside numbers are the same, then the product is defined. The outside numbers then give the size of the product.

$$\begin{array}{c} A \qquad B \qquad AB \\ m \times r \qquad r \times n = m \times n \\ \text{Inside} \\ \text{Outside} \end{array} \quad (3)$$

► **EXAMPLE 6 Determining Whether a Product Is Defined**

Suppose that  $A$ ,  $B$ , and  $C$  are matrices with the following sizes:

$$\begin{array}{ccc} A & B & C \\ 3 \times 4 & 4 \times 7 & 7 \times 3 \end{array}$$

Then by (3),  $AB$  is defined and is a  $3 \times 7$  matrix;  $BC$  is defined and is a  $4 \times 3$  matrix; and  $CA$  is defined and is a  $7 \times 4$  matrix. The products  $AC$ ,  $CB$ , and  $BA$  are all undefined. ◀

In general, if  $A = [a_{ij}]$  is an  $m \times r$  matrix and  $B = [b_{ij}]$  is an  $r \times n$  matrix, then, as illustrated by the shading in the following display,

$$AB = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1r} \\ a_{21} & a_{22} & \cdots & a_{2r} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ir} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mr} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2j} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ b_{r1} & b_{r2} & \cdots & b_{rj} & \cdots & b_{rn} \end{bmatrix} \quad (4)$$

the entry  $(AB)_{ij}$  in row  $i$  and column  $j$  of  $AB$  is given by

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \cdots + a_{ir}b_{rj} \quad (5)$$

Formula (5) is called the **row-column rule** for matrix multiplication.

## Matrix Products as Linear Combinations

**DEFINITION 6** If  $A_1, A_2, \dots, A_r$  are matrices of the same size, and if  $c_1, c_2, \dots, c_r$  are scalars, then an expression of the form

$$c_1 A_1 + c_2 A_2 + \dots + c_r A_r$$

is called a *linear combination* of  $A_1, A_2, \dots, A_r$  with *coefficients*  $c_1, c_2, \dots, c_r$ .

To see how matrix products can be viewed as linear combinations, let  $A$  be an  $m \times n$  matrix and  $\mathbf{x}$  an  $n \times 1$  column vector, say

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Then

$$\mathbf{Ax} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix} = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} \quad (10)$$

This proves the following theorem.

**THEOREM 1.3.1** *If  $A$  is an  $m \times n$  matrix, and if  $\mathbf{x}$  is an  $n \times 1$  column vector, then the product  $A\mathbf{x}$  can be expressed as a linear combination of the column vectors of  $A$  in which the coefficients are the entries of  $\mathbf{x}$ .*

► **EXAMPLE 8** Matrix Products as Linear Combinations

The matrix product

$$\begin{bmatrix} -1 & 3 & 2 \\ 1 & 2 & -3 \\ 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ -3 \end{bmatrix}$$

can be written as the following linear combination of column vectors:

$$2 \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} - 1 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ -3 \end{bmatrix}$$

### Transpose of a Matrix

**DEFINITION 7** If  $A$  is any  $m \times n$  matrix, then the *transpose of  $A$* , denoted by  $A^T$ , is defined to be the  $n \times m$  matrix that results by interchanging the rows and columns of  $A$ ; that is, the first column of  $A^T$  is the first row of  $A$ , the second column of  $A^T$  is the second row of  $A$ , and so forth.

#### ► EXAMPLE 11 Some Transposes

The following are some examples of matrices and their transposes.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 3 \\ 1 & 4 \\ 5 & 6 \end{bmatrix}, \quad C = [1 \ 3 \ 5], \quad D = [4]$$

$$A^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \\ a_{14} & a_{24} & a_{34} \end{bmatrix}, \quad B^T = \begin{bmatrix} 2 & 1 & 5 \\ 3 & 4 & 6 \end{bmatrix}, \quad C^T = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}, \quad D^T = [4] \quad \blacktriangleleft$$

Observe that not only are the columns of  $A^T$  the rows of  $A$ , but the rows of  $A^T$  are the columns of  $A$ . Thus the entry in row  $i$  and column  $j$  of  $A^T$  is the entry in row  $j$  and column  $i$  of  $A$ ; that is,

$$(A^T)_{ij} = (A)_{ji} \quad (14)$$

**THEOREM 1.4.8** *If the sizes of the matrices are such that the stated operations can be performed, then:*

(a)  $(A^T)^T = A$

(b)  $(A + B)^T = A^T + B^T$

(c)  $(A - B)^T = A^T - B^T$

(d)  $(kA)^T = kA^T$

(e)  $(AB)^T = B^T A^T$

*The transpose of a product of any number of matrices is the product of the transposes in the reverse order.*

*Trace of a Matrix*

**DEFINITION 8** If  $A$  is a square matrix, then the *trace of  $A$* , denoted by  $\text{tr}(A)$ , is defined to be the sum of the entries on the main diagonal of  $A$ . The trace of  $A$  is undefined if  $A$  is not a square matrix.

**▶ EXAMPLE 12 Trace**

The following are examples of matrices and their traces.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 2 & 7 & 0 \\ 3 & 5 & -8 & 4 \\ 1 & 2 & 7 & -3 \\ 4 & -2 & 1 & 0 \end{bmatrix}$$

$$\text{tr}(A) = a_{11} + a_{22} + a_{33}$$

$$\text{tr}(B) = -1 + 5 + 7 + 0 = 11 \quad \blacktriangleleft$$

In the exercises you will have some practice working with the transpose and trace operations.

### Exercise Set 1.3 (page 36)

1. (a) (b) (c)

3. (a) (d) (e) (h) (i) (j) (l)

5. (a) (b) (d) (h) (i) (l)

15.

23.

### True-False Exercises