$$\begin{bmatrix} 2 & 3 \\ 2 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & & \end{bmatrix} \quad \begin{bmatrix} 2+12 & 4+15 \\ 2 & 4 \end{bmatrix}$$

2nd Semester 1436/1437	**Final Exam**	**King Saud University**
(without calculators)	**Time allowed: 3 hours**	**College of Science**
Thursday 12-8-1437	**240 Math**	**Math. Department**

$B = I$

Q1: Let $B=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}, C=\begin{bmatrix} 5 & 2 & 0 \\ 4 & 1 & 4 \\ 0 & 2 & 3 \end{bmatrix}$ and $AB+AC=2I$, where $I=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

$AI + AC = 2I$

Find:

(a) A^{-1}

(b) det(A). $\boxed{A(B+C) = 2I}$ (5 marks)

Q2: Let V be the vector space **spanned** by the set S={ v_1=(1,-2,0), v_2=(3, -3,6), v_3=(-1, 5, 0), v_4=(2,2,4), v_5=(5,-7,12)}. Find a **subset** of S that forms a basis of V.

(4 marks)

$A(I + AC) = I$

Q3: Find a basis for each eigenspace of the matrix $A=\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$. Moreover, find the algebraic multiplicity and the geometric multiplicity of each eigenvalue of A and **deduce** if the matrix A is diagonalizable or not. (6 marks)

Q4: Let $\mathbb{R}^4$ be the Euclidean inner product space. Find the distance between the vectors **u**= (2,4,2,-2) and **v**= (-2,1,1,1). Also, show that these two vectors are orthogonal. (4 marks)

Q5: Assume that the vector space $\mathbb{R}^3$ has the Euclidean inner product. Apply the Gram-Schmidt process to transform the basis vectors (1,0,1), (0,1,2), (0,3,0) into an orthonormal basis. (6 marks)

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Q6: Let V be an inner product space, let v_o be any fixed vector in V, and let $T:V \to \mathbb{R}$ be the map defined by $T(v)=\langle v, v_o \rangle$ for all v in V. Show that:

(a) T is a linear transformation.

(b) If $v_o \in \ker(T)$, then $v_o = 0$. (5 marks)

Q7: Let $T:\mathbb{R}^3 \to \mathbb{R}^3$ be the linear operator defined by the formula:

$T(x_1,x_2,x_3)=(3x_1,-2x_1-4x_2,3x_1+4x_2-2x_3)$.

(a) Find $[T]_{S,B}$ where S is the standard basis of $\mathbb{R}^3$ and
B={v_1=(1,1,1),v_2=(1,1,0),v_3=(1,0,0)}.

(b) Show that T is one-to-one. (5 marks)

Q8: Prove the following statements:

(a) If $T:V \to W$ is a linear transformation, then the kernel of T is a subspace of V. (3 marks)

(b) If $T_1 : U \to V$ and $T_2 : V \to W$ are two linear transformations, then $(T_2 \circ T_1) : U \to W$ is also a linear transformation. (2 marks)

Q1 $B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $C = \begin{pmatrix} 5 & 2 & 0 \\ 4 & 1 & 4 \\ 0 & 2 & 3 \end{pmatrix}$ $\quad AB + AC = 2I$

Find a) A^{-1} $\quad$ b) $\det A$

a) $\because AB + AC = 2I \Rightarrow A(B+C) = 2I \xRightarrow{\times A^{-1}} A^{-1}A(B+C) = 2IA^{-1}$

$$B + C = 2A^{-1} \Rightarrow A^{-1} = \frac{1}{2}(B+C)$$

$$A^{-1} = \frac{1}{2}\left\{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 5 & 2 & 0 \\ 4 & 1 & 4 \\ 0 & 2 & 3 \end{pmatrix}\right\} = \frac{1}{2}\begin{pmatrix} 6 & 2 & 0 \\ 4 & 6 & 4 \\ 0 & 2 & 4 \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} 3 & 1 & 0 \\ 2 & 3 & 2 \\ 0 & 1 & 2 \end{pmatrix}$$

b) $$|A^{-1}| = \begin{vmatrix} 3 & 1 & 0 \\ 2 & 3 & 2 \\ 0 & 1 & 2 \end{vmatrix} \overset{-R_2+R_1}{=} \begin{vmatrix} 3 & 1 & 0 \\ 2 & 3 & 2 \\ -2 & -2 & 0 \end{vmatrix} = -2\begin{vmatrix} 3 & 1 \\ -2 & -2 \end{vmatrix}$$

$$= -2[-6+2] = -2(-4) = 8$$

$$|A^{-1}| = \frac{1}{|A|} \Rightarrow |A| = \frac{1}{8}.$$

I2 / Q_2 $v_1=(1,-2,0)$, $v_2=(3,-3,6)$, $v_3=(-1,5,0)$, $v_4=(2,2,4)$

$v_5=(5,-7,12)$

$$\begin{pmatrix} 1 & 3 & -1 & 2 & 5 \\ -2 & -3 & 5 & 2 & -7 \\ 0 & 6 & 0 & 4 & 12 \end{pmatrix} \xrightarrow{2R_1+R_2} \begin{pmatrix} 1 & 3 & -1 & 2 & 5 \\ 0 & 3 & 3 & 6 & 3 \\ 0 & 6 & 0 & 4 & 12 \end{pmatrix} \xrightarrow{\frac{1}{3}R_2}$$

$$\begin{pmatrix} 1 & 3 & -1 & 2 & 5 \\ 0 & 1 & 1 & 2 & 1 \\ 0 & 6 & 0 & 4 & 12 \end{pmatrix} \xrightarrow{-6R_2+R_3} \begin{pmatrix} 1 & 3 & -1 & 2 & 5 \\ 0 & 1 & 1 & 2 & 1 \\ 0 & 0 & -6 & -8 & 6 \end{pmatrix}$$

$$\begin{matrix} x & x & x \end{matrix}$$

The basis of V is $\{(1,-2,0), (3,-3,6), (-1,5,0)\}$.

T3/Q3 Eigenspace $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$

$$\lambda I - A = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} + \begin{pmatrix} -1 & -1 & 0 \\ 0 & -1 & 0 \\ -1 & -1 & -1 \end{pmatrix} = \begin{pmatrix} \lambda-1 & -1 & 0 \\ 0 & \lambda-1 & 0 \\ -1 & -1 & \lambda-1 \end{pmatrix} *$$

$$|\lambda I - A| = 0 \Rightarrow \begin{vmatrix} \lambda-1 & -1 & 0 \\ 0 & \lambda-1 & 0 \\ -1 & -1 & \lambda-1 \end{vmatrix} = 0 \Rightarrow (\lambda-1) \begin{vmatrix} \lambda-1 & -1 \\ 0 & \lambda-1 \end{vmatrix} = 0$$

$$\Rightarrow (\lambda-1)\left[(\lambda-1)^2 - 0\right] = 0 \Rightarrow (\lambda-1)^3 = 0$$

$$\Rightarrow \lambda = 1, \lambda = 1, \lambda = 1.$$

Eigen Value $\lambda = 1$ with algebraic multiplicity 3

Eigen vectors

* case $\lambda = 1$ $\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ -1 & -1 & 0 \end{pmatrix} \xrightarrow{-R_1+R_3} \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \xrightarrow[-R_3]{-R_1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$x_1 = 0$ and $x_2 = 0$ $x_3 = t$

$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ t \end{pmatrix} = t \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ $\lambda = 1$ has geometric multiplicity is 1

So A is diagonalizable because Algebric $\neq$ Geomtric
3 $\neq$ 1

Q4:- $u = (2,4,2,-2)$, $v = (-2,1,1,1)$

1) Find the distance between u, v

راجع 2) Show that u, v are orthogonal.

1) $d(u,v) = \|u - v\|$

$u - v = (2,4,2,-2) - (-2,1,1,1) = (2,4,2,-2) + (2,-1,-1,-1)$

$u - v = (4,3,1,-3)$

$\|u-v\|^2 = \langle (4,3,1,-3), (4,3,1,-3) \rangle$

$= 16 + 9 + 1 + 9 = 35$

$\therefore d(u,v) = \|u - v\| = \sqrt{35}$

2) $\langle u, v \rangle = \langle (2,4,2,-2), (-2,1,1,1) \rangle$

$= -4 + 4 + 2 - 2 = 0$

So u, v are orthogonal.

$\frac{5}{5}$

Q5 $v_1 = (1,0,1)$, $v_2 = (0,1,2)$, $v_3 = (0,3,0)$

$u_1 = v_1 = (1,0,1)$ $\Longrightarrow \|u_1\|^2 = \langle (1,0,1),(1,0,1)\rangle = 1+1 \cancel{+0} = 2$

$\|u_1\|^2 = 2$

$$u_2 = v_2 - \frac{\langle v_2, u_1\rangle}{\|u_1\|^2} u_1$$

$\langle v_2, u_1\rangle = \langle (0,1,2),(1,0,1)\rangle$

$= 0+0+2 = 2$

$$= (0,1,2) - \frac{\cancel{2}}{\cancel{2}}(1,0,1) = (0,1,2) + (-1,0,-1) = (-1,1,1)$$

$\|u_2\|^2 = \langle (-1,1,1),(-1,1,1)\rangle = 1+1+1 = 3.$

$$u_3 = v_3 - \frac{\langle v_3, u_1\rangle}{\|u_1\|^2} u_1 - \frac{\langle v_3, u_2\rangle}{\|u_2\|^2} u_2$$

$\langle v_3, u_1\rangle = \langle (0,3,0),(1,0,1)\rangle = 0$

$\langle v_3, u_2\rangle = \langle (0,3,0),(-1,1,1)\rangle = 3.$

$$u_3 = (0,3,0) - \frac{0}{2}u_1 - \frac{3}{3}(-1,1,1) = (0,3,0) + (1,-1,-1)$$

$u_3 = (1,2,-1)$ $\|u_3\|^2 = \langle (1,2,-1),(1,2,-1)\rangle$

$= 1+4+1 = 6$

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orthonormal basis is

$$\left\{ \frac{1}{\sqrt{2}}(1,0,1),\ \frac{1}{\sqrt{3}}(-1,1,1),\ \frac{1}{\sqrt{6}}(1,2,-1) \right\}$$

Q6) $T(v) = \langle v, v_0 \rangle$

(a) Show that T is Linear Transformation

اكمل (b) If $v_0 \in \ker T$ then $v_0 = 0$

(a) $T(u+v) = \langle u+v, v_0 \rangle = \langle u, v_0 \rangle + \langle v, v_0 \rangle = T(u) + T(v)$ $\quad \forall u, v \in V$

$T(\alpha u) = \langle \alpha u, v_0 \rangle = \alpha \langle u, v_0 \rangle = \alpha T(u)$ $\quad \alpha \in \mathbb{R}$.

(b) $v_0 \in \ker T \Rightarrow T(v_0) = \langle v_0, v_0 \rangle = 0 \Leftrightarrow v_0 = 0$

من تعريف الضرب الداخلي.

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(Q7) $T(x_1, x_2, x_3) = (3x_1, -2x_1 - 4x_2, 3x_1 + 4x_2 - 2x_3)$

(a) Find $[T]_{S,B}$ $\quad$ S is standard basis of $\mathbb{R}^3$

$B = \{ v_1 = (1,1,1),\ v_2 = (1,1,0),\ v_3 = (1,0,0) \}$.

$S = \{ (1,0,0)\ (0,1,0)\ (0,0,1) \}$.

$T(1,0,0) = (3, -2, 3)$

$T(0,1,0) = (0, -4, 4)$ $\quad$ $T(0,0,1) = (0, 0, -2)$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 3 & 0 & 0 \\ 1 & 1 & 0 & -2 & -4 & 0 \\ 1 & 0 & 0 & 3 & 4 & -2 \end{array}\right) \xrightarrow{-R_2+R_1} \left(\begin{array}{ccc|ccc} 0 & 0 & 1 & 5 & 4 & 0 \\ 1 & 1 & 0 & -2 & -4 & 0 \\ 1 & 0 & 0 & 3 & 4 & -2 \end{array}\right)$$

$$\xrightarrow{-R_3+R_2} \left(\begin{array}{ccc|ccc} 0 & 0 & 1 & 5 & 4 & 0 \\ 0 & 1 & 0 & -5 & -8 & 2 \\ 1 & 0 & 0 & 3 & 4 & -2 \end{array}\right) \xRightarrow{R_3 \leftrightarrow R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 4 & -2 \\ 0 & 1 & 0 & -5 & -8 & 2 \\ 0 & 0 & 1 & 5 & 4 & 0 \end{array}\right)$$

B

$$[T]_{S,B} = \begin{pmatrix} 3 & 4 & -2 \\ -5 & -8 & 2 \\ 5 & 4 & 0 \end{pmatrix}$$

b) Show that T is one-one.

$$|[T]| = \begin{vmatrix} 3 & 4 & -2 \\ -5 & -8 & 2 \\ 5 & 4 & 0 \end{vmatrix} \overset{R_1+R_2}{=} \begin{vmatrix} 3 & 4 & -2 \\ -2 & -4 & 0 \\ 5 & 4 & 0 \end{vmatrix}$$

$$= -2 \begin{vmatrix} -2 & -4 \\ 5 & 4 \end{vmatrix} = -2(-8+20) = -24 \neq 0$$

so T is one-one.