King Saud University,
College of Sciences
Mathematical Department.

Mid-Term 2/Summer/2013
Full Mark:25. Time 1H30mn
14/9/1434

Question 1[5] Consider the initial value problem

$$\begin{cases} \sqrt{16-x^2}.y'' + 2(\ln x).y' + y = x^2 \\ y(2) = 1, \ y'(2) = 3. \end{cases} \qquad (*)$$

Find the largest interval for which the initial value problem (*) has a unique solution.

Question 2[5] If $y_1 = x^3$ is a solution for the homogeneous equation

$$x^2y'' - 3xy' + 3y = 0,$$

in the interval $(0, \infty)$, then find its second solution y_2.

Question3[5] By sing the method of variation of parameters, solve the differential equation

$$y'' + y' - 2y = 5e^x$$

Question4[5] By using the undetermined coefficients method, give only the form of the particular solution y_p of the two differential equation

$$y^{(3)} - 2y'' = e^{2x}\cos x - xe^{2x}\sin x$$

Question 5[5]. Solve the following linear system of differential equations

$$\begin{cases} \frac{d^2y}{dt^2} = x \\ \frac{dx}{dt} = 4\frac{dy}{dt} \end{cases}$$

Answer Sheet.

MidTerm 2 Math 204

Q1: $a_2(x)$ is Continuous for $x \in [-4, 4]$

$a_1(x) = 2\ln x$ is Cont for $x > 0$ ($x \in (0, \infty)$)

$a_0(x) = 1$ is Cont for $x \in \mathbb{R}$.

$f(x) = x^2$ " "

All functions are Continuous for $x \in (0, 4)$

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$a_2(x) \neq 0$ for $x \neq \pm 4$

Since $x_0 = 2 \in (0, 4)$, Then the largest interval for the given I.V.P has a unique solution is $I = (0, 4)$

Q2 If we use the formula $y_2 = y_1 \int \frac{e^{-\int p\,dx}}{y_1^2}\,dx$, we have

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$$y_2 = x^3 \int \frac{e^{+3\int \frac{x}{x^2}dx}}{x^6}\,dx = x^3 \int \frac{e^{3\ln x}}{x^6}\,dx = x^3 \int \frac{dx}{x^3}$$

$$= \frac{-x}{2}$$

(*) We can also use the reduction of order method

Q3: $y'' + y' - 2y = 5e^x$

$y_g = y_c + y_p$

Charact Eq: $m^2 + m - 2 = 0 \Rightarrow m_1 = 1, \quad m_2 = -2$

2½

$y_c = c_1 e^x + c_2 e^{-2x}$

$y_p = C_1(x)\, e^x + C_2(x)\, e^{-2x}$

$$\begin{cases} C_1'(x)e^{x} + C_2'(x)e^{-2x} = 0 \\ C_1'(x)e^{x} - 2C_2'(x)e^{-2x} = 5e^{x} \end{cases}$$

$$D = \begin{vmatrix} e^{x} & e^{-2x} \\ e^{x} & -2e^{-2x} \end{vmatrix} = -3e^{-x} \neq 0 \quad \forall x \in \mathbb{R}$$

$$C_1'(x) = \frac{\begin{vmatrix} 0 & e^{-2x} \\ 5e^{x} & -2e^{-2x} \end{vmatrix}}{-3e^{-x}} = \frac{-5e^{-x}}{-3e^{-x}} = \frac{5}{3}$$

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$$\Rightarrow C_1(x) = \frac{5}{3}x$$

$$C_2'(x) = \frac{\begin{vmatrix} e^{x} & 0 \\ e^{x} & 5e^{x} \end{vmatrix}}{-3e^{-x}} = \frac{5e^{2x}}{-3e^{-x}} = -\frac{5}{3}e^{3x}$$

$$\Rightarrow C_2(x) = -\frac{5}{9}e^{3x}$$

Thus $y_p(x) = \frac{5}{3}x e^{x} - \frac{5}{9}e^{x} = \left(\frac{5}{3}x - \frac{5}{9}\right)e^{x}$

Q4 : $y^{(3)} - 2y'' = e^{2x}\cos x - x e^{2x}\sin x$

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$m^3 - 2m^2 = 0 \Rightarrow m_1 = m_2 = 0,\ m_3 = 2$

$y_p = x^s\left[(Ax+B)e^{2x}\cos x + (Cx+D)e^{2x}\sin x\right]$

$\alpha + i\beta = 2 + i$ is not a root for the ch Eq

$\Rightarrow s = 0$

Hence $y_p = (Ax+B)e^{2x}\cos x + (Cx+D)e^{2x}\sin x$

Q_5 : $\begin{cases} y'' = x \\ x' = 4y' \end{cases}$ operator form: $\begin{cases} D^2[y] - x = 0 & \to (1) \\ D[x] - 4D[y] = 0 & \to (2) \end{cases}$

$D = \frac{d}{dt}$

1

We apply D to (1) and take the sum, we get

$$y^{(3)} - 4y' = 0$$

2

Charact Eq: $m^3 - 4m = 0 \Rightarrow m(m^2 - 4) = 0$

$\Rightarrow m_1 = 0,\ m_2 = 2,\ m_3 = -2$

$\boxed{y(t) = c_1 + c_2 e^{2t} + c_3 e^{-2t}}$

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Now $y' = 2c_2 e^{2t} - 2c_3 e^{-2t}$

$y'' = 4c_2 e^{2t} + 4c_3 e^{-2t}$

Hence $\boxed{x(t) = 4c_2 e^{2t} + 4c_3 e^{-2t}}$

$$\begin{cases} x(t) = 4c_2 e^{2t} + 4c_3 e^{-2t} \\ y(t) = c_1 + c_2 e^{2t} + c_3 e^{-2t} \end{cases} ;\ c_1, c_2, c_3 \in \mathbb{R}$$