King Saud University,
College of Sciences
Mathematical Department.

Mid-Term 2/S1/2012
Full Mark:25. Time 1H30mn
04/12/2012

Question 1[**5,4**]. **a)** Given the initial value problem

$$\begin{cases} \sqrt{x-1}.y'' + \frac{1}{x-3}y' + xy = 1 \\ y(2) = 0, \ y'(2) = 1. \end{cases} \qquad (*)$$

Find the largest interval for which the initial value problem (*) has a unique solution.

b) Determine whether the functions: $f(x) = x$, $g(x) = x\ln x$, $h(x) = x^2 \ln x$ are linearly dependent or linearly independent on $(0, \infty)$.

Question 2 [**5**] If $y_1 = e^x$ is a solution for the homogeneous equation

$$xy'' - 2(x+1)y' + (x+2)y = 0, \qquad x > 0$$

then by using the method of reduction of order find the general solution of the nonhomogeneous equation

$$xy'' - 2(x+1)y' + (x+2)y = xe^x$$

Question 3[**5**]. Solve the nonhomogeneous differential equation

$$y'' + 4y' + 4y = e^{-2x}.$$

Question 4[**6**]. Solve the following linear system of differential equations

$$\begin{cases} y' - x + 2y = t \\ x' + y = 1. \end{cases}$$

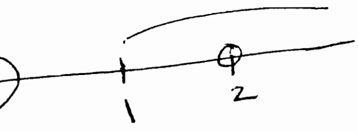
Answer Sheet

Exam 2 / S1 / 2011

Answer to Q1

a) $a_2(x) = x-1$, $a_1(x) = -x$, $a_0(x) = 1$, $f(x) = 1$ are continuous on $\mathbb{R} = (-\infty, \infty)$. 3

$a_2(x) = x-1 = 0 \Rightarrow x = 1$.



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Since $x_0 = 2$, then the largest interval for which the given I.V.P. (*) has a unique solution is $(1, 3)$

b) $y = e^{x} u(x) \Rightarrow y' = e^{x}(u'+u)$, $y'' = e^{x}(u''+2u'+u)$.

Thus $(x-1)y'' - xy' + y = (x-1)e^{x}(u''+2u'+u) - x e^{x}(u'+u) + e^{x}u = 0$ 2

$\Leftrightarrow (x-1)u'' + \underbrace{(2x-2-x)}_{(x-2)}u' = 0$, then

let $v = u'$, then

$(x-1)v' + (x-2)v = 0 \Rightarrow \dfrac{dv}{v} = \dfrac{2-x}{x-1}\,dx$

$\Rightarrow \ln\left|\dfrac{v}{c}\right| = -(x - \ln|x-1|)$ 2

$\Rightarrow v(x) = c_1 e^{-x}(x-1)$

Hence $u' = c_1 e^{-x}(x-1) \Rightarrow u(x) = c_1\int (x-1)e^{-x}dx + c_2$

$\Rightarrow u(x) = -xe^{-x} \Rightarrow \boxed{y_2 = -x}$ with $c_1 = 1$, $c_2 = 0$ 2

Answer to Q2

a) $y'' + 16y = 0$

Charact Eq: $m^2 + 16 \Rightarrow m_1 = 4i,\ m_2 = -4i$ (2)

$y_{gh} = C_1 \cos 4t + C_2 \sin 4t$ (2)

$y(0) = 1 \Rightarrow 1 = C_1$

$y' = -4C_1 \sin 4t + 4C_2 \cos 4t$

$y'(0) = 8 \Rightarrow 4C_2 = 8 \Rightarrow C_2 = 2$ (2)

Thus $y_p = \cos 4t + 2 \sin 4t$

b) Charact Eq: $m^3 - 3m^2 + 3m - 1 = 0,\ m_1 = 1$ (2)

$m^3 - 3m^2 + 3m - 1 = (m-1)(m^2 - 2m + 1) = 0$

$m^2 - 2m + 1 = 0 \Leftrightarrow (m-1)^2 = 0 \Rightarrow m_2 = m_3 = 1$

$m = 1$ is a root of order of multiplicity 3 (triple root)

Thus $y_p = (Ax + B) + x^{\cancel{s}\,3}(Cx^2 + Dx + E)e^x$ (4)

$s = 3$ Since $r = 1$ is a triple root.

Hence $y_p = (Ax + B) + (Cx^5 + Dx^4 + Ex^3)e^x$

Answer to Q3 $\quad x^2y'' - xy' + y = 2x, \quad x>0$ (Cauchy Euler Eq)

$y = x^m$

Charact Eq $\quad m^2 - 2m + 1 = 0 \Rightarrow (m-1)^2 = 0$

$y_h = C_1 x + C_2 x \ln x$ $\quad$ 2

Using the method of variation of parameters, we have

with $y_p = C_1(x)x + C_2(x) x \ln x$

$$\begin{cases} C_1' x + C_2' x \ln x = 0 \\ C_1' + C_2' (\ln x + 1) = \frac{2}{x} \end{cases}$$

$$\Delta = W = \begin{vmatrix} x & x\ln x \\ 1 & \ln x + 1 \end{vmatrix} = x$$

$$C_1'(x) = \frac{\begin{vmatrix} 0 & x\ln x \\ \frac{2}{x} & \ln x + 1 \end{vmatrix}}{x} = -\frac{2\ln x}{x}$$ $\quad$ 2

$\Rightarrow C_1(x) = -2\int \frac{\ln x}{x}\,dx$

Let $u = \ln x \Rightarrow du = \frac{dx}{x}$

$C_1(x) = -2\int u\,du = \frac{-2u^2}{2} = -u^2 = -(\ln x)^2$

$$C_2'(x) = \frac{\begin{vmatrix} x & 0 \\ 1 & \frac{2}{x} \end{vmatrix}}{x} = \frac{2}{x} \Rightarrow C_2(x) = 2\ln x$$ $\quad$ 2

Hence $y_p = -(\ln x)^2 x + 2x(\ln x)^2 = x(\ln x)^2$

$y_g = y_h + y_p = C_1 x + C_2 x \ln x + x(\ln x)^2$ $\quad$ 1

Answer to Q4 : $\begin{cases} 4y'' - 4x = 1 \\ x'' - y = t^3 \end{cases}$

operator form $\begin{cases} 4D^2[y] - 4x = 1 \longrightarrow (1) \\ D^2[x] - y = t^3 \longrightarrow (2) \end{cases}$ 2

We apply $4D^2$ to (2), we get

$4D^4[x] - 4D^2[y] = 24t \longrightarrow (3)$

(1) + (3) yields 2

$4x^{(4)} - 4x = 1 + 24t \longrightarrow (3)$

Ch Eq: $4m^4 - 4 = 0 \Rightarrow m^4 - 1 = 0 = (m^2-1)(m^2+1) \Rightarrow$

$m_1 = 1,\ m_2 = -1,\ m_3 = i,\ m_4 = -i$ 1

Hence $x_{gh}(t) = C_1 e^t + C_2 e^{-t} + C_3 \cos t + C_4 \sin t$

$x_p = At + B,\ x_p' = A,\ x_p'' = 0$

Hence $-4At - 4B = 1 + 24t$ 2

$\Rightarrow A = -6,\ B = -\frac{1}{4}$

$\Rightarrow x_p(t) = -6t - \frac{1}{4}$

Hence $x(t) = C_1 e^t + C_2 e^{-t} + C_3 \cos t + C_4 \sin t - 6t - \frac{1}{4}$

$x'(t) = C_1 e^t - C_2 e^{-t} - C_3 \sin t + C_4 \cos t - 6$

$x''(t) = C_1 e^t + C_2 e^{-t} - C_3 \cos t - C_4 \sin t$ 2

Thus $y(t) = C_1 e^t + C_2 e^{-t} - C_3 \cos t - C_4 \sin t - t^3$

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