KING SAUD UNIVERSITY
DEPARTMENT OF MATHEMATICS
TIME: 1H 30 mn, FULL MARKS: 40, 22/01/1432
MATH 204

Question 1. **a)** [7] Find the largest interval for which the following initial value problem has a unique solution

$$\begin{cases} x^2y'' + \frac{x}{\sqrt{2-x}}y' + \frac{2}{\sqrt{x}}y = 0 \\ y(1) = 2, \quad y'(1) = 1 \end{cases}$$

b) [7] If the function $y_1 = e^{-x}$ is a solution of the differential equation

$$(x^2+2)y'' - 2xy' - (x^2+2x+2)y = 0.$$

Use formula to find the second solution y_2, and hence find the general solution.

Question 2. **a)** [9] Find the general solution of the nonhomogeneous differential equation:

$$y'' + y' - 2y = \cos x.$$

b) [6]. Determine only the form of the particular solution of the diffrenetial equation

$$y'' + 2y' + 10y = x^2e^{-x}\sin 3x.$$

Question 3. [9] By using the transformation $x = e^t$, find the general solution of the Cauchy-Euler equation

$$x^2y'' + xy' + 4y = \cos(\ln x^2), \quad x > 0.$$

Question 4. [9] Solve the system of differential equations

$$\begin{cases} \frac{dx}{dt} - 4x - y = 2 \\ \frac{dy}{dt} + 2x - y = 0. \end{cases}$$

NOTE: For question 1, answer either part **a)** or part **b)**.

Answear Sheet

Midsterm 2

Q2: a) $y''+y'-2y=\cos x \rightarrow (*)$

$y_g = y_{gh} + y_p$

$y_{gh} = ?$ $\quad y''+y'-2y=0$, characteristic equation $m^2+m-2=0$ (2)

$\iff m_1 = 1,\ m_2 = -2$

$y_{gh} = C_1 e^{x} + C_2 e^{-2x}$ (2)

$y_p = x^s A\cos x + x^s B\sin x = A\cos x + B\sin x$ ($s=0$ since $m=0$ is not a sol for the char eq) (3)

$y_p' = -A\sin x + B\cos x\ ,\quad y_p'' = -A\cos x - B\sin x$

By substitution in the DE $\rightarrow (*)$ and identification we get

$$\begin{cases} -3A+B=1 \\ A-3B=0 \end{cases} \Rightarrow A = \frac{3}{10}\ ,\ B = \frac{1}{10}$$

Hence $y_g = C_1 e^{x} + C_2 e^{-2x} - \frac{3}{10}\cos x + \frac{1}{10}\sin x$ (?)

b) $y''+2y'+10y = x^2 e^{-x}\sin 3x$

Routes of Char Eq: $m^2+2m+10=0 \iff \Delta = 4-40=-36<0$ (1)

$m_1 = \frac{-2+6i}{2} = -1+3i$ (1)

$m_2 = -1-3i$

$S=1$ Since $\alpha+i\beta = -1+3i$ is a root for the charact Eq; thus

$y_p = x\left[a_1x^2+b_1x+c_1\right]e^{-x}\cos 3x + x\left[a_2x^2+b_2x+c_2\right]e^{-x}\sin 3x$

$= \left[\left(a_1x^3+b_1x^2+c_1\right)\cos x + \left(a_2x^3+b_2x^2+c_2x\right)\sin x\right]e^{-x}$ (4)

Q1: a) $\begin{cases} x^2 y'' + \frac{x}{\sqrt{2-x}} y' + \frac{2}{\sqrt{x}} y = 0 \\ y(1) = 2, \quad y'(1) = 1 \end{cases}$

$a(x) = x^2$, $b(x) = \frac{x}{\sqrt{2-x}}$, $c(x) = \frac{2}{\sqrt{x}}$ are continuous on $(0,2)$ and $a(x) = x^2 \neq 0$ for $x \neq 0$

Since $x_0 = 1 \in (0,2)$, the largest interval for which the I.V.P has a unique solution is $(0,2)$.

b) $y_2 = e^{-x} \int \frac{e^{-\int \frac{-2x}{x^2+2} dx}}{e^{-2x}} dx = e^{-x} \int \frac{e^{\ln(x^2+2)}}{e^{-2x}} dx$

$= e^{-x} \int \frac{x^2+2}{e^{-2x}} dx = e^{-x} \int x^2 e^{2x} dx + 2e^{-x} \int e^{2x} dx$

$= \frac{e^{x}}{4} (2x^2 - 2x + 5)$

$y_g = C_1 e^{-x} + C_2 e^{x} (2x^2 - 2x + 5)$

Q4 $\begin{cases} x' - 4x - y = 2 \\ y' + 2x - y = 0 \end{cases}$

operator form: $\begin{cases} (D-4)[x] - y = 2 \longrightarrow (1) \\ (D-1)[y] + 2x = 0 \longrightarrow (2) \end{cases}$

To eliminate x, we apply $(D-4)$ to (2) and multiply (1) by -2 and then sum to get

$(D-4)(D-1)[y] + 2y = -4$

$\Leftrightarrow \quad y'' - 5y' + 6y = -4$

$y_g = y_{gh} + y_p$

Ch Eq: $m^2 - 5m + 6 = 0 \Rightarrow m_1 = 2, \ m_2 = 3$

$y_{gh} = C_1 e^{2t} + C_2 e^{3t}$

$y_p = A$, $y_p' = 0$, $y_p'' = 0$, $\quad 6A = -4 \Rightarrow A = -\frac{2}{3}$

$y_p = -\frac{2}{3}$, $\quad y_g = C_1 e^{2t} + C_2 e^{3t} - \frac{2}{3}$

$x_g(t) = \frac{y_g - y_g'}{2} = -\frac{C_1}{2} e^{2t} - C_2 e^{3t} - \frac{1}{3}$.

Q_3: $x^2y''+xy'+4y=\cos(\ln x^2) \longrightarrow (1)$, $x>0$

Let $x=e^t \Rightarrow t=\ln x$ and $dx=e^t dt$

We have $x\frac{dy}{dx}=\frac{dy}{dt}$, $x^2\frac{d^2y}{dx^2}=\frac{d^2y}{dt^2}-\frac{dy}{dt}$.

The DE takes the form

$$\frac{d^2y}{dt^2}+4y=\cos 2t \longrightarrow (2)$$

The charact Eq for the homog Eq:

$m^2+4=0 \Rightarrow m_1=2i,\ m_2=-2i$

$y_{gh}=C_1\cos 2t+C_2\sin 2t$

$y_p=C_1(t)\cos 2t+C_2(t)\sin 2t$

$$\begin{cases} C_1'(t)\cos 2t+C_2'(t)\sin 2t=0 \\ -2C_1'(t)\sin 2t+2C_2'(t)\cos 2t=\cos 2t \end{cases}$$

$$\Delta=\begin{vmatrix}\cos 2t & \sin 2t\\ -2\sin 2t & 2\cos 2t\end{vmatrix}=2$$

$$C_1'(t)=\frac{\begin{vmatrix}0 & \sin 2t\\ \cos 2t & 2\cos 2t\end{vmatrix}}{2}=-\frac{1}{2}\sin 2t\cos 2t=-\frac{1}{4}\sin 2t\, d(\sin 2t)$$

$$\Rightarrow C_1(t)=-\frac{1}{4}\frac{\sin^2 2t}{2}=-\frac{1}{8}\sin^2 2t$$

$$C_2'(t)=\frac{\begin{vmatrix}\cos 2t & 0\\ -2\sin 2t & \cos 2t\end{vmatrix}}{2}=\frac{\cos^2 2t}{2}\Rightarrow C_2(t)=\frac{1}{2}\int\frac{1+\cos 4t}{2}dt$$

$$\Rightarrow C_2(t)=\frac{t}{4}+\frac{1}{16}\sin 4t$$

Thus $y_p=-\frac{1}{8}\sin^2 2t\cos 2t+\left(\frac{t}{4}+\frac{1}{16}\sin 4t\right)\sin 2t$

Converting back to x, we have

$$y_g=C_1\cos(2\ln x)+C_2\sin(2\ln x)-\frac{1}{8}\sin^2(2\ln x)\cos(2\ln x)+\left(\frac{\ln x}{4}+\frac{1}{16}\sin(4\ln x)\right)\sin(2\ln x).$$