King Saud University,
College of Sciences
Mathematical Department.

Mid-Term1 /S2/2014
Full Mark:25. Time 1H30mn
14/05/1435

**Question 1[4,4].** **a)** Find and sketch the largest region of the $xy-$plane for which the initial value problem

$$\begin{cases} \frac{(x-1)}{\sqrt{x+3}}dy + x\ln(y-2)dx = 0 \\ y(-2) = 3. \end{cases}$$

has a unique solution.

**b)** Find a curve having the slope:

$$\frac{(1+y^2)\cos x}{2y(1+(\sin x)^2)}$$

and passes through the point $(\pi/2, 0)$.

**Question 2[4,4].** **a)** Solve the initial value problem

$$\begin{cases} [2x(\sin(x^2)\ln y]dx - \frac{\cos(x^2)}{y}dy = 0, \quad y > 0 \\ y(0) = e \end{cases}$$

**b)** Find the genral solution of the differential equation

$$(\frac{y}{2\ln x} + y^2)dx - xdy = 0, \quad x > 0.$$

**Question 3[4].** Solve the differential equation

$$\frac{dy}{dx} = \frac{y}{x\ln y - x\ln x + x}, \qquad x > 0, \; y > 0.$$

**Question 4[7].** A hot iron rod was left in a room with temperature equals to $T_s$. After one minute the temperature of the rod is equal to $2T_s$. Afetr two minutes it is equal to $\frac{3}{2}T_s$. What was the initial temperature of the rod.

**Question 5[5].** Find the family of orthogonal trajectories for the family of curves

$$y^2 = Cx^3 - 2.$$

**Remark:** Answer question 4 or question 5.

Ex1

Consider the IVP. $\boxed{1}$

Q, 9
$$\begin{cases} \dfrac{(x-1)}{\sqrt{x+3}}\,dy + x\ln(y-2)\,dx = 0 \\ y(-2) = 3 \end{cases}$$

Find the largest region on which the IVP has a solution

Solution

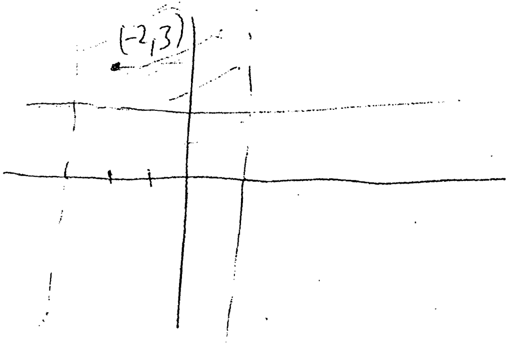
$$\frac{dy}{dx} = \frac{-x\sqrt{x+3}\,\ln(y-2)}{x-1}$$

is continuous on $D = \{(x,y)\ \ x \geq -3,\ x \neq 1,\ y > 2\}$ 1

$$\frac{\partial f}{\partial y} = \frac{-x\sqrt{x+3}}{(x-1)(y-2)}$$ is continuous on

$$D_1 = \{(x,y),\ x \geq -3\ \ x \neq 1,\ y \neq 2\}$$ 1

$f$ and $\frac{\partial f}{\partial y}$ are continuous on $D \cap D_1 = D$




the largest region on which the IVP has a solution is

$$D_2 = \{(x,y),\ -3 < x < 1,\ y > 2\}$$ 2

**Q1 : b)** $\dfrac{dy}{dx} = \dfrac{(1+y^2)\cos x}{2y(1+\sin^2 x)}$   ( Sep Eq )

We have $$\frac{2y}{1+y^2}\,dy = \frac{\cos x}{1+\sin^2 x}\,dx$$

By integration :

$$\ln(1+y^2) = \int \frac{\cos x}{1+\sin^2 x}\,dx \qquad 1$$

let $t = \sin x \Rightarrow dt = \cos x\,dx$ , then

$$\ln(1+y^2) = \int \frac{dt}{1+t^2} = \tan^{-1} t = \tan^{-1}(\sin x) + C_1$$

Hence $1+y^2 = e^{C_1}\exp\{\tan^{-1}(\sin x)\} = C\exp\{\tan^{-1}(\sin x)\}$

where $C = e^{C_1}$   1

Since $y(\frac{\pi}{2}) = 0$, Then $1 = C\exp\{\tan^{-1} 1\} = C e^{\frac{\pi}{4}}$

$\Rightarrow C = e^{-\frac{\pi}{4}}$.

Thus $1+y^2 = e^{-\frac{\pi}{4}}\exp\{\tan^{-1}(\sin x)\}$.   2

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**Q2 a)** $\dfrac{\partial M(x,y)}{\partial y} = \dfrac{2x\sin(x^2)}{y}$, $\dfrac{\partial N}{\partial x} = \dfrac{2x\sin(x^2)}{y}$ $\Rightarrow \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x} \Rightarrow$ the DE is exact   1

Then exists $F(x,y)$ /
$$\begin{cases} \dfrac{\partial F}{\partial x} = 2x\sin(x^2)\ln y \longrightarrow (1) \\ \dfrac{\partial F}{\partial y} = -\dfrac{\cos(x^2)}{y} \longrightarrow (2) \end{cases}$$

From (2), we have $F(x,y) = -\cos(x^2)\ln y + A(x) \rightarrow (3)$

From (3), $\dfrac{\partial F}{\partial x} = 2x\sin(x^2)\ln y + A'(x) \rightarrow (4)$   1

From (4) and (1) it follows that

$$A'(x) = 0 \Longrightarrow A(x) = C_1$$

Hence $F(x,y) = -\cos(x^2)\ln y + C_1 = C$

$$\Rightarrow \cos(x^2)\ln y = C$$

$$y(0) = e \Longrightarrow C = 1$$

Hence $\cos(x^2)\ln y = 1$

other method: $d\left(\ln y \cdot \cos(x^2)\right) = 0$

$$\Rightarrow \ln y \cdot \cos(x^2) = C$$

$$y(0) = e$$

$$\Longrightarrow C = 1$$

$$\Rightarrow \cos(x^2)\ln y = 1$$

Q2 b) $x\,dy = \left(\frac{y}{2\ln x} + y^2\right)dx$

$\Rightarrow \quad x y' - \frac{y}{2\ln x} = y^2$

$\Rightarrow \quad y' - \frac{1}{2x\ln x} y = \frac{y^2}{x}$

Dividing by $y^2$ to have

$y' y^{-2} - \frac{1}{2x\ln x} y^{-1} = \frac{1}{x}$

Let $u = y^{-1} \Rightarrow u' = -y^{-2} y'$

Hence we have:

$+u' + \frac{u}{2x\ln x} = -\frac{1}{x}$ (LE)

$\mu(x) = e^{\frac{1}{2}\int \frac{dx}{x\ln x}} = e^{\frac{1}{2}\ln \ln x} = (\ln x)^{\frac{1}{2}}$

Hence we have

$$\frac{d}{dx}\left(u.(\ln x)^{\frac{1}{2}}\right) = -\frac{(\ln x)^{\frac{1}{2}}}{x}$$

$\Rightarrow \quad u.(\ln x)^{\frac{1}{2}} = -\int \frac{(\ln x)^{\frac{1}{2}}}{x} dx$

Let $\ln x = t \Rightarrow \frac{dx}{x} = dt$

Hence $u(\ln x)^{\frac{1}{2}} = -\int \sqrt{t}\,dt + c = -\frac{2}{3}(\ln x)^{\frac{3}{2}} + c$

$\Rightarrow \quad \frac{1}{y}(\ln x)^{\frac{1}{2}} = -\frac{2}{3}(\ln x)^{\frac{3}{2}} + c$

$\Rightarrow \quad \frac{1}{y} = -\frac{2}{3}\ln x + c(\ln x)^{-\frac{1}{2}}$

$\boxed{5}$

$$y' = \frac{y}{x(\ln\frac{y}{x}+1)} = \frac{y}{x}\cdot\frac{1}{\ln\frac{y}{x}+1}$$ (齐次)

Let $u = \frac{y}{x} \Rightarrow y' = xu' + u$

Hence $xu' + u = u \cdot \frac{1}{\ln u + 1}$ 1

$$\Rightarrow xu' = \frac{u}{\ln u + 1} - u = \frac{-u\ln u}{\ln u + 1}$$

$$\Rightarrow \frac{(1+\ln u)\,du}{u\ln u} = -\frac{dx}{x}$$

$$\Rightarrow \int\frac{du}{u\ln u} + \int\frac{du}{u} = -\ln|x| + C$$ 1

Let $t = \ln u \Rightarrow dt = \frac{du}{u}$

$$\int\frac{dt}{t} + \ln|u| = -\ln|x| + C$$

$$\Rightarrow \ln|\ln u| + \ln|u| = -\ln|x| + C$$

$$\Rightarrow \ln\left|\ln\frac{y}{x}\right| + \ln\left|\frac{y}{x}\right| + \ln|x| = C$$

$$\ln\left|\ln\frac{y}{x}\right| + \ln|y| = C$$

$$\Rightarrow \ln\left|\frac{\ln\frac{y}{x}}{y^{-1}}\right| = C.$$

$$\Rightarrow y\ln\left|\frac{y}{x}\right| = \pm e^{C} = C_1.$$

$\Rightarrow$

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Q4: $\frac{dT}{dt} = k(T - T_s)$

$\Rightarrow \frac{dT}{T - T_s} = k\,dt \Rightarrow T(t) = T_s + Ce^{kt}$ 1

$T(1) = 2T_s \Rightarrow T_s + Ce^{k} = 2T_s \Rightarrow T_s = Ce^{k} \longrightarrow (1)$ 1

$T(2) = \frac{3}{2}T_s \Rightarrow T_s + Ce^{2k} = \frac{3}{2}T_s \Rightarrow \frac{T_s}{2} = Ce^{2k} \longrightarrow (2)$ 1

(1) and (2) imply $\frac{1}{2} = e^{k} \longrightarrow (3)$

From (1) $C = T_s e^{-k}$, then from (3);

$C = 2T_s$ 2

Hence $T(t) = T_s + 2T_s\left(\frac{1}{2}\right)^t$

$T(0) = T_s + 2T_s = 3T_s$ 2

$\underline{Q_5}$. $y^2 = c x^3 - 2$

$2yy' = 3cx^2 = 3x^2\left(\frac{y^2+2}{x^3}\right)$ $\quad x \neq 0$

$2yy' = 3\left(\frac{y^2+2}{x}\right)$

$\Rightarrow \quad y' = \frac{3}{2}\left(\frac{y^2+2}{xy}\right)$ $\quad$ DE associated to the F.C.

The DE corresponding to the F. O.T

$$y' = -\frac{2}{3}\frac{xy}{(y^2+2)}$$

$\Leftrightarrow \quad \frac{y^2+2}{y}\,dy = -\frac{2}{3}x\,dx$

$\Rightarrow \quad \frac{y^2}{2} + 2\ln|y| = -\frac{x^2}{3} + C$

$\Rightarrow \quad \frac{3}{2}y^2 + \ln(y^6) + x^2 = C$