King Saud University,
College of Sciences
Mathematical Department.

Mid-Term2 /S2/2016
Full marks:25. Time 1H30mn
05/07/1437

Question 1[4pts,4pts]. **a)** Find the largest interval on which the initial value problem

$$\begin{cases} (x^2 - 3x - 4)y'' + \sqrt{9 - x^2}y' + 9xy = 0 \\ y(0) = 1, \ y'(0) = 0 \end{cases}$$

has a unique solution.

b) If $y_1 = x - 1$ is a solution of the differential equation

$$(x^2 - 2x - 1)y'' - 2(x - 1)y' + 2y = 0,$$

find its general solution.

Question 2[4pts,4pts]. **a)** Use the method of undetermined coefficients to write down the form of y_p for the differential equation

$$y''' + 9y' = 4 - x + e^{2x}\cos x + \sin 3x.$$

Do not determine the constants.

b) Solve the initial value problem

$$\begin{cases} x^2y'' - xy' + y = x^2 \\ y(1) = 1, \ y'(1) = 0. \end{cases} \quad x > 0,$$

Question 3[4pts]. Find the general solution of the differential equation

$$y'' - y' = xe^{-2x}$$

Question 4[5pts]. Solve the system of differential equations

$$\begin{cases} x' - x = y \\ y' - y = x + 1 \end{cases}$$

Q:1 (a) $a_2(x) = x^2 - 3x - 4$ Cont. on $(-\infty, \infty)$

$a_1(x) = \sqrt{9 - x^2}$ Cont on $[-3, 3]$

$a_0(x) = 9x$ cont. on $(-\infty, \infty)$

$\Rightarrow a_2(x), a_1(x)$, and $a_0(x)$ Cont on $[-3, 3]$ 1

& $a_2(x) = 0$ at $(x-4)(x+1) = 0$

i.e $x = 4, \; x = -1$ 1

$-3 \quad -1 \quad 3$

1 1

$y(0) = 1 \Rightarrow x = 0, y = 1$

$y'(0) = 0 \Rightarrow x = 0, y' = 0$

$\Rightarrow x_0 = 0 \in (-1, 3)$

$\Rightarrow$ required interval $(-1, 3)$

(b) $(x^2 - 2x - 1)y'' - 2(x-1)y' + 2y = 0$ 1

$$P(x) = \frac{-2x+2}{x^2 - 2x - 1} \Rightarrow -\int P\,dx = \int \frac{2x-2}{x^2-2x-1}\,dx$$

$$= \ln|x^2 - 2x - 1|$$

$$\Rightarrow e^{-\int P\,dx} = x^2 - 2x - 1$$

$\Rightarrow$ 2nd Solution is $= (x-1) \int \frac{x^2 - 2x - 1}{x^2 - 2x + 1}\,dx$

$$= (x-1)\left[\int \left(1 - \frac{2}{(x-1)^2}\right) dx\right]$$

$$= (x-1)\left[x + \frac{2}{(x-1)}\right]$$ 2

$$= x(x-1) + 2$$

$x^2-2x+1 \,\overline{)\, x^2 - 2x - 1}$ quotient 1; $x^2 - 2x + 1$; remainder -2

$\Rightarrow$ G.S $= c_1(x-1) + c_2(x^2 - x + 2)$ 1

Q:2 (a) $y''' + 9y' = 4 - x + e^{2x}\cos x + \sin 3x$

Homog part $y''' + 9y' = 0$

$m^3 + 9m = 0 \Rightarrow m(m^2+9) = 0$ (1)

$\Rightarrow m = 0$ & $m = \pm i3$

$\Rightarrow y_c = c_1 + c_2\cos 3x + c_3 \sin 3x$ (1)

But $f(x) = 4 - x + e^{2x}\cos x + \sin 3x$

$\Rightarrow y_p = x(Ax+B) + e^{2x}[C\cos x + D\sin x]$ (2)

$+ x(E\cos 3x + F\sin 3x)$

(b) $x^2y'' - xy' + y = x^2, \quad x > 0$

$y(1) = 1, \ y'(1) = 0$

Chauchy Euler equ.

$x^2y'' - xy' + y = 0$ Put $y = x^m$

$m(m-1) - m \cancel{+} 1 = 0$

$m^2 - 2m + 1 = 0 \Rightarrow (m-1)^2 = 0 \Rightarrow m = 1, 1$

$\Rightarrow y_c = c_1 x + c_2 x\ln x \quad \Rightarrow y_1 = x, \ y_2 = x\ln x$

& $f(x) = \frac{x^2}{x^2} = 1$

$y_p = u_1 x + u_2 x\ln x$

$W = \begin{vmatrix} x & x\ln x \\ 1 & 1+\ln x \end{vmatrix} = x + x\cancel{\ln x} - x\cancel{\ln x} = x$

$W_1 = \begin{vmatrix} 0 & x\ln x \\ 1 & 1+\ln x \end{vmatrix} = -x\ln x$

$W_2 = \begin{vmatrix} x & 0 \\ 1 & 1 \end{vmatrix} = x$

$u_1' = \frac{W_1}{W} = -\ln x$

$\Rightarrow \boxed{u_1 = -x\ln x + x}$

$u_2' = \frac{W_2}{W} = \frac{x}{x} = 1$

$\Rightarrow \boxed{u_2 = x}$

$$\Rightarrow y_p = (-x\ln x + x)x + x(x\ln x) = x^2$$

$$\Rightarrow y = y_c + y_p = c_1 x + c_2 x\ln x + x^2 \quad \text{———} \quad (*)$$

$$\boxed{\begin{aligned} &y(1) = 1 \Rightarrow x = 1, y = 1 \\ &1 = c_1 + 1 \Rightarrow c_1 = 0 \end{aligned}}$$

$$\boxed{\begin{aligned} &\left.\begin{aligned} &y'(1) = 0 \\ &x = 1, y' = 0 \end{aligned}\right] \quad \begin{aligned} &y' = c_1 + c_2[1 + \ln x] + 2x \\ &0 = 0 + c_2[1 + 0] + 2 \end{aligned} \\ &\Rightarrow c_2 + 2 = 0 \Rightarrow c_2 = -2 \end{aligned}}$$

$$\Rightarrow y = -2x\ln x + x^2$$

Q: 3

$$y'' - y' = x e^{-2x} \quad \text{———} \quad (1)$$

$$y'' - y' = 0 \Rightarrow m^2 - m = 0 \Rightarrow m(m-1) = 0, \quad m = 0, 1$$

$$\Rightarrow y_c = c_1 + c_2 e^{x}$$

Now $y_p = (Ax + B)e^{-2x}$, $y_p' = Ae^{-2x} - 2(Ax+B)e^{-2x}$

$$y_p'' = -2Ae^{-2x} - 2Ae^{-2x} + 4(Ax+B)e^{-2x}$$

$(1) \Rightarrow$

$$-4Ae^{-2x} + 4Axe^{-2x} + 4Be^{-2x} - Ae^{-2x} + 2Axe^{-2x} + 2Be^{-2x} = xe^{-2x}$$

$$(-5A - 2B)e^{-2x} + 6Axe^{-2x} = xe^{-2x}$$

$$\Rightarrow +6A = 1 \Rightarrow A = 1/6$$

$$-4A + 4B - A + 2B = 0 \Rightarrow -5A + 6B = 0 \Rightarrow B = 5/36$$

$y = c_1 + c_2 e^{x} + \left(\frac{1}{6}x + \frac{5}{36}\right)e^{-2x}$ ##

Q. 4

$$\left.\begin{aligned} x' - x &= y \\ y' - y &= x + 1 \end{aligned}\right\} \Rightarrow \begin{aligned} (D-1)x - y &= 0 \\ -x + (D-1)y &= 1 \end{aligned} \quad X(D-1$$

$(D-1)^2 x - x = 1$

$(D^2 - 2D + 1)x - x = 1 \Rightarrow \boxed{x'' - 2x' = 1}$ — ①

$m^2 - 2m = 0 \Rightarrow m = 0,\ m = 2$

$\boxed{x_c = c_1 + c_2 e^{2t}}$ $\quad x_p = At$

$x_p' = A,\ x_p'' = 0$

In ① $0 - 2A = 1 \Rightarrow A = -\frac{1}{2} \Rightarrow \boxed{x_p = -\frac{t}{2}}$

$\Rightarrow \boxed{x = x_c + x_p = c_1 + c_2 e^{2t} - \frac{t}{2}}$ — ②

Put in $y = x' - x = 2c_2 e^{2t} - \frac{1}{2} - c_1 - c_2 e^{2t} + \frac{t}{2}$

$\boxed{y = c_2 e^{2t} - c_1 + \frac{1}{2}(t-1)}$