King Saud University, College of Sciences Mathematical Department.

Math 204 Mid-Term 2 /2015-2016
Full Mark: 25.
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Question 1.[5]. Find the largest interval for which the following initial value problem has a unique solution

$$\begin{cases} x^2y'' + \frac{x}{\sqrt{2-x}}y' + \frac{2}{\sqrt{x}}y = 0, \\ \quad y(1) = 0, \quad y'(1) = 1. \end{cases}$$

Question 2.[5,5]. **a)** Solve the Cauchy-Euler equation

$$x^2y'' + xy' = 12\ln x, \qquad x > 0.$$

b) If $y_1 = \frac{\cos x}{x}$ is a solution of the differential equation

$$xy'' + 2y' + xy = 0,$$

use the formula to find the second solution y_2 and hence find the general solution of the nonhomogeneous equation

$$xy'' + 2y' + xy = 1,$$

by using the variation of parameters method.

Question 3.[5]. By employing the undetermined coefficients method, find a particular solution of the differential equation

$$y'' + 2y' + y = 4x^2 - 3.$$

Question 4.[5]. Solve the system of linear differential equations

$$\begin{cases} \frac{dx}{dt} + \frac{dy}{dt} + 2y = 0 \\ \frac{dx}{dt} - 3x - 2y = 0 \end{cases}$$

Q_1 : $a_2(x) = x^2$ is continuous on $\mathbb{R} = (-\infty, \infty)$ and $a_2(x) \neq 0$ $\forall x \neq 0$

$a_1(x) = \dfrac{x}{\sqrt{2-x}}$ is continuous on $(-\infty, 2)$, ①

$a_0(x) = \dfrac{2}{\sqrt{x}}$ is continuous on $(0, \infty)$ ①

The functions $a_2(x), a_1(x), a_0(x)$ are continuous on

$\mathbb{R} \cap (-\infty, 2) \cap (0, \infty) = (0, 2)$ which ①

contains $x_0 = 1$ and $a_2(x) \neq 0$.

We can take any interval $I \subset (0, 2)$ such that

$x_0 = 1 \in I$, but I is the largest ②

interval for which the given I.V.P has a unique solution

$\underline{Q_2}$ a) $x^2y'' + xy' = 12\ln x \quad , x > 0$

Characteristic equation: $m^2 = 0 \Rightarrow m_1 = m_2 = 0 \Rightarrow$

$y_1 = x^0 = 1$

$y_2 = \int e^{-\int \frac{dx}{x}} dx = \int e^{-\ln x} dx$ (2)

$= \int \frac{dx}{x} = \ln x$

Hence $y_h = C_1 + C_2 \ln x$

To find y_p, we use the Variation of parameters method

$y_p = C_1(x) + C_2(x)\ln x$

$$\begin{cases} C_1' + C_2' \ln x = 0 \\ \dfrac{C_2'}{x} = \dfrac{12\ln x}{x^2} \end{cases}$$ (3)

$\Rightarrow C_2'(x) = \dfrac{12\ln x}{x} \Rightarrow C_2(x) = 12\int \ln x \, d(\ln x)$

$= 12\dfrac{(\ln x)^2}{2} = 6(\ln x)^2$

$C_1' = -C_2' \ln x = -\dfrac{12\ln x}{x} \cdot \ln x$

$\Rightarrow C_1(x) = -12 \int \ln^2 x \, d(\ln x)$

$= -\dfrac{12}{3}(\ln x)^3 = -4(\ln x)^3$

Thus $y_p = -4(\ln x)^3 + 6(\ln x)^3 = 2(\ln x)^3$

$$\boxed{y_g = C_1 + C_2 \ln x + 2(\ln x)^3}$$

Q_2 b) $y_2 = y_1 \int \frac{e^{-\int p\,dx}}{y_1^2}\, dx = \frac{\cos x}{x} \int \frac{x^2 e^{-\int \frac{2}{x}dx}}{\cos^2 x}\, dx$

$$= \frac{\cos x}{x} \int \frac{x^2}{x^2} \frac{dx}{\cos^2 x} = \frac{\cos x}{x} \int \sec^2 x\, dx$$

$$= \frac{\cos x}{x} \int d(\tan x)$$

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$$= \frac{\cos x}{x} \cdot \frac{\sin x}{\cos x} = \frac{\sin x}{x}$$

$$y_{gh} = C_1 \frac{\cos x}{x} + C_2 \frac{\sin x}{x}$$

$y_p = C_1(x)\, y_1 + C_2(x)\, y_2$, where

$$\begin{cases} C_1'(x) \frac{\cos x}{x} + C_2'(x) \frac{\sin x}{x} = 0 \\ -C_1'(x)\left(\frac{\sin x}{x} + \frac{\cos x}{x^2}\right) + C_2'(x)\left(\frac{\cos x}{x} - \frac{\sin x}{x^2}\right) = \frac{1}{x} \end{cases}$$

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$$\Delta = \begin{vmatrix} \frac{\cos x}{x} & \frac{\sin x}{x} \\ -\frac{\sin x}{x} - \frac{\cos x}{x^2} & \frac{\cos x}{x} - \frac{\sin x}{x^2} \end{vmatrix} = \frac{1}{x^2}$$

$$C_1'(x) = \frac{\begin{vmatrix} 0 & \frac{\sin x}{x} \\ \frac{1}{x} & \frac{\cos x}{x} - \frac{\sin x}{x^2} \end{vmatrix}}{\frac{1}{x^2}} = -\sin x$$

$$\Rightarrow C_1(x) = +\cos x$$

$$C_2'(x) = \frac{\begin{vmatrix} \frac{\cos x}{x} & 0 \\ -\frac{\sin x}{x} - \frac{\cos x}{x^2} & \frac{1}{x} \end{vmatrix}}{\frac{1}{x^2}} = \cos x \Rightarrow C_2(x) = \sin x$$

$$\Rightarrow y_p = \frac{\cos^2 x}{x} + \frac{\sin^2 x}{x} = \frac{1}{x}$$

$$\Rightarrow y_g = C_1 \frac{\cos x}{x} + C_2 \frac{\sin x}{x} + \frac{1}{x}.$$

Q3

$$y'' + 2y' + y = 4x^2 - 3$$

$$y_g = y_h + y_p \qquad 1$$

charact Eq: $m^2 + 2m + 1 = 0 \Rightarrow m_1 = m_2 = -1$

$$\Rightarrow y_h = C_1 e^{-x} + C_2 x e^{-x}$$

$$y_p = Ax^2 + Bx + C, \quad y_p' = 2Ax + B, \quad y_p'' = 2A$$

Hence $Ax^2 + (4A+B)x + (2A+2B+C) = 4x^2 - 3 \qquad 2$

$$\Rightarrow A = 4, \quad 4A + B = 0 \Rightarrow B = -16$$

$$C = -3 - 2A - 2B = 21$$

So $y_p = 4x^2 - 16x + 21 \qquad 2$

Q4: operator form

$$\begin{cases} D[x] + (D+2)[y] = 0 & \longrightarrow (1) \\ (D-3)[x] - 2y = 0 & \longrightarrow (2) \end{cases}$$

To eliminate x, we apply $(D-3)$ to (1) and $-D$ to (2) $\qquad 1$

We get

$$\begin{cases} (D-3)D[x] + (D-3)(D+2)[y] = 0 \\ -D(D-3)[x] + 2D[y] = 0 \end{cases}$$

$$\Rightarrow y'' + y' - 6y = 0$$

$$\Rightarrow m^2 + m - 6 = 0 \Rightarrow m_1 = 2, \quad m_2 = -3 \qquad 2$$

$$\Rightarrow \boxed{y(t) = C_1 e^{2t} + C_2 e^{-3t}}$$

From (1), we have $x' = -2C_1 e^{2t} + 3C_2 e^{-3t} - 2C_1 e^{2t} - 2C_2 e^{-3t}$

$$= -4C_1 e^{2t} + C_2 e^{-3t}$$

$$\Rightarrow x(t) = -2C_1 e^{2t} - \frac{C_2}{3} e^{-3t} + C_3 \qquad 2$$

Eq (2) gives $-4C_1 e^{2t} + C_2 e^{-3t} + 6C_1 e^{2t} + C_2 e^{-3t} - 3C_3$

$$- 2C_1 e^{2t} - 2C_2 e^{-3t} = 0$$

$$\Rightarrow C_3 = 0 \Rightarrow \boxed{x(t) = -2C_1 e^{2t} - \frac{C_2}{3} e^{-3t}}$$