King Saud University,
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Mid-Term 2/S2/2014
Full Mark: 26. Time 1H30mn
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Question 1[4,3]. **a)** Consider the initial value problem

$$\begin{cases} (x+2)y'' + \frac{x}{\sqrt{3-x}}y' + 2y\ln(x+5) = 0 \\ y(1) = 0,\ y'(1) = 1. \end{cases} \qquad (*)$$

Find the largest positive value of the constant α so that the initial value problem (*) has a unique solution for all $x \in I = (-\alpha, \alpha)$.

b) Find the general solution of the differential equation

$$xy'' + (x-1)y' + (3-12x)y = 0, \quad x > 0,$$

if $y = e^{3x}$ solves the differential equation.

Question 2[5,5]. **a)** Find the constants a, b, c, d and e such that the differential equation

$$ay^{(4)} + by^{(3)} + cy'' + dy' + ey = 0,$$

has the solutions: $y_1 = xe^{-x}$ and $y_2 = e^{2x}\cos 3x$.

b) Solve the nonhomogeneous differential equation

$$y^{(4)} + y^{(3)} = 1 - e^{-x}.$$

Question 3[4,5]. **a)** Consider the differential equation

$$y'' - \frac{2}{x^2}y = 0, \quad x > 0.$$

What is the type of this equation ?. Solve it by using the substitution $x = e^t$.

b) Solve the following linear system of differential equations. Find t for which $x(t) = y(t)$.

$$\begin{cases} \frac{d^2y}{dt^2} + \frac{dx}{dt} + x = 0 \\ \frac{dy}{dt} - y + x = \sin t \end{cases}$$

Q1 a) $a_2(x) = x+2$ is Cont for all $x \in \mathbb{R}$

$a_1(x) = \dfrac{x}{\sqrt{3-x}}$ is Cont for $x \in (-\infty, 3)$

$a_0(x) = 2\ln(x+5)$ is Cont for $x \in (-5, \infty)$

$a_2(x) = x+2 \neq 0$ if $x \neq -2$

$x_0 = 1 \in (-2, 3)$ where all functions are Cont and $a_2(x) \neq 0$

$-5 \quad -2 \quad 0 \quad 1 \quad 3$

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Thus the largest interval for which the I.V.P has a unique solution is $x \in (-2, 3)$

$I = (-\alpha, \alpha) = (-2, 2)$. 1

b) $xy'' + (x-1)y' + (3-12x)y = 0$, $x > 0$

If we use the formula $y_2 = y_1 \displaystyle\int \frac{e^{-\int P(x)\,dx}}{y_1^2}\,dx$

$y'' + \dfrac{x-1}{x}\,y' + \left(\dfrac{3-12x}{x}\right)y = 0$ 1

$P(x) = \dfrac{x-1}{x}$

$y_2 = y_1 \displaystyle\int \frac{e^{-\int (1-\frac{1}{x})\,dx}}{e^{6x}}\,dx = y_1 \int x e^{-7x}\,dx$ 1

$= e^{3x}\left[-\frac{1}{7}xe^{-7x} - \frac{1}{49}e^{-7x}\right]$

$= -e^{-4x}\left(\frac{x}{7} + \frac{1}{49}\right)$ 1

$\Rightarrow y = c_1y_1 + c_2y_2 = c_1e^{3x} + c_2e^{-4x}\left(x + \frac{1}{7}\right)$

Q2 : a) The solutions of the DE should be

$y_1 = e^{-x}$, $y_2 = xe^{-x}$, $y_3 = e^{2x}\cos 3x$, $y_4 = e^{2x}\sin 3x$ 1

The roots of the characteristic equation are

$m_1 = -1$, $m_2 = -1$, $m_3 = 2+3i$, $m_4 = 2-3i$ 1

The characteristic equation is then:

$(m+1)(m+1)(m-2-3i)(m-2+3i) = 0$ 1

$\Rightarrow (m+1)^2\left[(m-2)^2 + 9\right] = 0$

$\Rightarrow (m^2+2m+1)(m^2-4m+13] = 0$

$\Rightarrow m^4 - 2m^3 + 6m^2 + 22m + 13 = 0$ 1

$\Rightarrow a = 1$, $b = -2$, $c = 6$, $d = 22$, $e = 13$ 1

$y^{(4)} - 2y^{(3)} + 6y'' + 22y' + 13y = 0$

b) charact eq $m^4 + m^3 = 0 \Rightarrow m^3(m+1) = 0$

$m_1 = 0$, $m_2 = 0$, $m_3 = 0$, $m_4 = -1$ 1

$y_c = C_1 + C_2 x + C_3 x^2 + C_4 e^{-x}$ 1

$y_p = y_{p_1} + y_{p_2} = Ax^3 + Bxe^{-x}$ 1

$y'_p = 3Ax^2 - Bxe^{-x} + Be^{-x}$

$y''_p = 6Ax + Bxe^{-x} - Be^{-x} - Be^{-x} = 6Ax + Bxe^{-x} - 2Be^{-x}$

$y^{(3)}_p = 6A + Be^{-x} - Bxe^{-x} + 2Be^{-x} = 6A - Bxe^{-x} + 3Be^{-x}$

$y^{(4)}_p = -Be^{-x} + Bxe^{-x} - 3Be^{-x}$

$\Rightarrow \cancel{-4Be^{-x}} + \cancel{Bxe^{-x}} + 6A - \cancel{Bxe^{-x}} + \cancel{3Be^{-x}} = 1 - e^{-x}$ 1

$\Rightarrow -Be^{-x} + 6A = 1 - e^{-x} \Rightarrow A = \frac{1}{6}$, $B = 1$

$\Rightarrow \quad y_p = \frac{x^3}{6} + x e^{-x}$ 1

Hence $\quad y_g = y_c + y_p = C_1 + C_2 x + C_3 x^2 + C_4 e^{-x} + \frac{1}{6} x^3 + x e^{-x}$

Q_3: a) $y'' - \frac{2}{x^2} y = 0 \quad x > 0$

$\Rightarrow x^2 y'' - 2y = 0$ (*) (Cauchy - Euler Eq)

$x = e^t \qquad \frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} = e^t \frac{dy}{dx} = x \frac{dy}{dx}$

$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(x \frac{dy}{dx}\right) = x \frac{d}{dx}\left(x \frac{dy}{dx}\right)$ 1

$= x \frac{dy}{dx} + x^2 \frac{d^2y}{dx^2} = \frac{dy}{dt} + x^2 \frac{d^2y}{dx^2}$

Hence the DE (*) becomes:

$\frac{d^2y}{dt^2} - \frac{dy}{dt} - 2y = 0$ 1

Charact Eq: $m^2 - m - 2 = 0$

$\Delta = 1 + 8 = 9$

$m_1 = 2, \quad m_2 = -1$ 2

$y_1 = e^{2t} = e^{2 \ln x} = x^2$

$y_2 = e^{-t} = e^{-\ln x} = \frac{1}{x}$

$y = C_1 x^2 + C_2 \frac{1}{x}$

b) $\begin{cases} y'' + x' + x = 0 \\ y' - y + x = \sin t \end{cases}$

Operator form $\begin{cases} (D+1)[x] + D^2[y] = 0 \longrightarrow (1) \\ x + (D-1)[y] = \sin t \longrightarrow (2) \end{cases}$ 1

To eliminate x, we apply $(D+1)$ to (2) and multiply (1) by -1 and sum, we get

$$-(D+1)[x] - D^2[y] = 0 \longrightarrow (3)$$ 1

$$(D+1)[x] + (D+1)(D-1)[y] = (D+1)[\sin t] \longrightarrow (4)$$

$$(3)+(4) \Rightarrow -D^2[y] + D^2[y] - y = \cos t + \sin t$$ 1

$$\Rightarrow y = -\cos t - \sin t$$

$$y' = \sin t - \cos t$$

Hence $x = \cancel{\sin t} - \cancel{\cos t} - \cancel{\sin t} - \sin t + \cancel{\cos t}$ 1

$$x(t) = -\sin t$$

We see that $x - y = \cos t$

$x = y$ iff $t = \frac{\pi}{2} + k\pi$, $k = 0, \pm 1, \pm 2, \ldots$ 1