King Saud University,
College of Sciences
Mathematical Department.

Mid-Term 2/S1/2018
Full Mark: 25. Time 1H30mn
15/11/2018

Question 1[4,4] **a)** Find the largest interval for which the following initial value problem has a unique solution

$$\begin{cases} (x-2)y'' + 3y = x \\ y(0) = 0, \ y'(0) = 1. \end{cases}$$

b) Solve the nonhomogeneous differential equation

$$y'' - y = 2e^x - 2x^2 + 5$$

Question 2 [4,3]. **a)** If $y_1 = e^x$ is a solution of the differential equation

$$y'' + 3y' - 4y = x,$$

then use reduction of order method to obtain its general solution.

b) Determine a homogeneous linear differential equation with constant coefficients having the fundamental set of solutions:

$$y_1 = 7, \quad y_2 = 8x, \quad y_3 = e^{-x}\cos x, \quad y_4 = e^{-x}\sin x, \quad y_5 = 5x^2.$$

Question 3 [5] Find the general solution of the differential equation

$$xy'' - 2y' + \frac{2}{x}y = 3x^3 + 2x; \quad x > 0.$$

Question 4 [5] Solve the following linear system of differential equations.

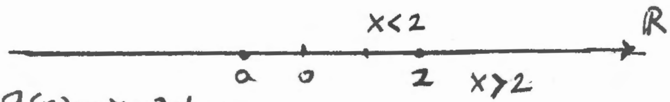
$$\begin{cases} 16x'' - y = 0 \\ y'' - 16x = 32t \end{cases}$$

Solutions Complete of Mid-Exam

M 204, First Semester. 1439/1440H

Question 1

(a) $\begin{cases} (x-2)y'' + 3y = x \\ y(0)=0,\ y'(0)=1 \end{cases}$



$a_2(x) = x-2 \neq 0$

$a_2(x) = x-2$, $a_1(x)=0$, $a_0(x)=3$ and $g(x)=x$ are continuous on $\mathbb{R}$. (1)

But $a_2(x) \neq 0$ if $x \in (-\infty, 2)$ or $x \in (2, \infty)$. As $0 \in (-\infty, 2)$,

then the largest interval I for which the IVP has (1)

a unique solution is $I = (-\infty, 2)$. (2)

Note also that the IVP has a unique solution on $(-2,2)$, centred at $x_0 = 0$.

(b) $y'' - y = -2x^2 + 2e^x + 5$

1) We find the solution of $y''-y=0$, $m^2-1=0$, $m=\mp 1$, then

$$y_c = c_1 e^{x} + c_2 e^{-x} \quad (1)$$

2) $y_p = (Ax^2+Bx+C) + Dxe^x$ (1)

$$y_p' = 2Ax + B + De^x + Dxe^x, \quad y_p'' = 2A + 2De^x + Dxe^x$$

$$y_p'' - y_p = 2A + 2De^x + \cancel{Dxe^x} - Ax^2 - Bx - C - \cancel{Dxe^x} = -2x^2 + 2e^x + 5$$

$$2A - C = 5, \quad -A = -2, \quad 2D = 2, \quad B = 0$$

$$\boxed{A=2,\ C=-1,\ D=1,\ B=0}$$

Then $\boxed{y_p = 2x^2 - 1 + xe^x}$ (1), and

$$\boxed{y = y_c + y_p = c_1 e^x + c_2 e^{-x} + 2x^2 - 1 + xe^x} \quad (1)$$

Question (2)

(a) $y'' + 3y' - 4y = x$, $y_1 = e^x$ is a given solution

We put $y = uy_1 = e^x u$

$y' = e^x u + e^x u'$, $y'' = e^x u + 2e^x u' + e^x u''$

$y'' + 3y' - 4y = \cancel{e^x u} + 2e^x u' + e^x u'' + \cancel{3e^x u} + 3e^x u' - \cancel{4e^x u}$
$= x$

$e^x u'' + 5e^x u' = x$ or $u'' + 5u' = xe^{-x}$

Let $w = u'$, $w' = u''$

$w' + 5w = xe^{-x}$, $\mu(x) = e^{\int 5dx} = e^{5x}$ (1)

$we^{5x} = \int xe^{-x} \cdot e^{5x} dx = \int xe^{4x} dx$

$we^{5x} = \frac{1}{4}xe^{4x} - \int \frac{1}{4}e^{4x} dx = \frac{1}{4}xe^{4x} - \frac{1}{16}e^{4x} + c_1$

$u' = w = \frac{1}{4}xe^{-x} - \frac{1}{16}e^{-x} + c_1 e^{-5x}$ (0)

$u = \int [\frac{1}{4}xe^{-x} - \frac{1}{16}e^{-x} + c_1 e^{-5x}] dx$

$u = \frac{1}{4}[-xe^{-x} - e^{-x}] + \frac{1}{16}e^{-x} - \frac{c_1}{5}e^{-5x} + c_2$

$u = -\frac{x}{4}e^{-x} + (\frac{-1}{4} + \frac{1}{16})e^{-x} - \frac{c_1}{5}e^{-5x} + c_2$

$\boxed{u = -\frac{x}{4}e^{-x} - \frac{3}{16}e^{-x} - \frac{c_1}{5}e^{-5x} + c_2}$ (2)

$y = e^x u = -\frac{x}{4} - \frac{3}{16} - \frac{c_1}{5}e^{-5x} + c_2 e^x$

$\boxed{y = -\frac{x}{4} - \frac{3}{16} + c_3 e^{-4x} + c_2 e^x}$, $c_3 = \frac{-c_1}{5}$

ⓑ The D.Eq has a general solution :

$$y_G = c_1(7) + c_2(8x) + c_3(5x^2) + c_4 e^{-x}\cos x + c_5 e^{-x}\sin x$$

$$y_G = c_1^* + c_2^* x + c_3^* x^2 + c_4 e^{-x}\cos x + c_5 e^{-x}\sin x .$$

1

So the roots of characteristic equation are

$m_1 = 0, \quad m_2 = 0, \quad m_3 = 0, \quad m_4 = -1+i, \quad m_5 = -1-i$

Then the charateristic equation is

1

$$m^3(m-(-1+i))(m-(-1-i)) = 0$$

$$m^3((m+1)+i)((m+1)-i) = 0$$

$$m^3((m+1)^2+1) = \boxed{m^3(m^2+2m+2) = 0}$$

2

$m^5 + 2m^4 + 2m^3 = 0$, hence the D.E is

$$\boxed{y^{(5)} + 2y^{(4)} + 2y''' = 0}$$

Question ③

$$xy'' - 2y' + \frac{2}{x}y = 3x^3 + 2x ; \qquad x>0$$

$$x^2y'' - 2xy' + 2y = 3x^4 + 2x^2 ; \qquad x>0$$

1) $x^2y'' - 2xy' + 2y = 0, \quad y = x^m, \quad m(m-1) - 2m + 2 = 0$

$m^2 - 3m + 2 = (m-1)(m-2) = 0 , \quad m=1, \ m=2$

$y_c = c_1 x + c_2 x^2 , \quad y_1 = x, \quad y_2 = x^2$

1

2) $y_p = y_1 u_1 + y_2 u_2$ s.t. $xu_1' + x^2 u_2' = 0$

$u_1' + 2x u_2' = 3x^2 + 2$

$W = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = x^2, \quad W_1 = \begin{vmatrix} 0 & x^2 \\ 3x^2+2, & 2x \end{vmatrix} = -3x^4 - 2x^2$

$$u_1' = \frac{W_1}{W} = \frac{-3x^4-2x^2}{x^2} = -3x^2-2, \quad \boxed{u_1 = -x^3-2x}$$

$$W_2 = \begin{vmatrix} x & 0 \\ 1 & 3x^2+2 \end{vmatrix} = 3x^3+2x, \quad u_2' = \frac{W_2}{W} = \frac{3x^3+2x}{x^2}$$

$$u_2' = 3x + \frac{2}{x}, \quad \boxed{u_2 = \frac{3}{2}x^2 + 2\ln x}$$

$$y_p = xu_1 + x^2u_2 = -x^4 - 2x^2 + \frac{3}{2}x^4 + 2x^2\ln x$$

$$\boxed{y_p = \frac{1}{2}x^4 - 2x^2 + 2x^2\ln x}$$

$$\boxed{y = y_c + y_p = c_1x + c_2x^2 + \frac{1}{2}x^4 - 2x^2 + 2x^2\ln x}$$ is the general

Solution of the D.E.

Question 4

$$\begin{cases} 16\ddot{x} - y = 0 \\ \ddot{y} - 16x = 32t \end{cases} \Rightarrow \begin{matrix} 16D^2x - y = 0 \\ -16x + D^2y = 32t \end{matrix}, \text{ then}$$

$$\begin{array}{l} 16D^4x - D^2y = 0 \\ -16x + D^2y = 32t \\ \hline + \end{array} \qquad 16x^{(4)}(t) - 16x = 32t$$

$$x^{(4)} - x = 2t$$

1) $x^{(4)} - x = 0, \quad x = e^{mt}, \qquad (m-1)(m+1)(m+i)(m-i) = 0$

$$m = 1, \quad m = -1, \quad m = i, \quad m = -i$$

$$\boxed{x_c(t) = c_1e^t + c_2e^{-t} + c_3\cos t + c_4\sin t}$$

$$x_p = At + B, \quad x_p = -2t$$

$$\boxed{x(t) = x_c + x_p = c_1e^t + c_2e^{-t} + c_3\cos t + c_4\sin t - 2t}$$

But $y = 16\ddot{x}$, then $\dot{x} = c_1e^t - c_2e^{-t} - c_3\sin t + c_4\cos t - 2$

$$\ddot{x} = c_1e^t + c_2e^{-t} - c_3\cos t - c_4\sin t$$

$$\boxed{y(t) = 16c_1e^t + 16c_2e^{-t} - 16c_3\cos t - 16c_4\sin t}$$