King Saud University.
College of Sciences
Mathematical Department.

Mid-Term1 /S2/2017
Full Mark:25. Time 1H30mn
13/07/1438

Question 1[4,4]. **a)** Determine the largest local region in the $xy-$plane for which the following differential equation

$$(x^2 - x - 6)\frac{dy}{dx} = \ln(4 - y^2),$$

would have a unique solution through the point $(1, 1)$.

b) Solve the differential equation:

$$x\ln(x+1)\frac{dy}{dx} - 2x + \frac{xy}{x+1} + y\ln(x+1) = 0, \quad x > -1.$$

Question 2[4,4]. **a)** Solve the following differential equation by using a suitable integrating factor

$$(xy + 1)dx + (x^2y + x^2/2 + 2x)dy = 0.$$

b) Write the differential equation in the form of Bernoulli's equation, hence solve it

$$(xy^3 - y^3 - x^2e^x)dx + 3xy^2dy = 0, \quad x > 0, \ y \neq 0.$$

Question 3[4]. Solve the differential equation

$$\left(x - y\tan^{-1}(\frac{y}{x})\right)dx + x\tan^{-1}(\frac{y}{x})dy = 0, \quad x > 0.$$

Question 4[5]. A cake is removed from a 350^0F oven and placed to cool in a room with temperature 75^0F. In 15 minutes the pie has a temperature of 150^0F. Determine the time required to cool the cake to a temperature of 80^0F so that it may be eaten.

Answer Sheet
MIDI / S2 / M204 (2017)

Q1 a) : $f(x,y) = \dfrac{\ln(4-y^2)}{(x-3)(x+2)}$, $\dfrac{\partial f}{\partial y} = \dfrac{-2y}{(4-y^2)(x-3)(x+2)}$

f is Continuous on $D_1 = \{(x,y) \in \mathbb{R}^2 ; \ |y|<2, \ x \neq 3, \ x \neq -2\}$ (1)

$\dfrac{\partial f}{\partial y}$ is Continuous on $D_2 = \{(x,y) \in \mathbb{R}^2 : \ |y| \neq 2 , \ x \neq 3, \ x \neq -2\}$ (1)

Both f and $\dfrac{\partial f}{\partial y}$ are continuous on $D_1 \cap D_2 = D_1$

Since $(1,1) \in D_3 = \{(x,y) \in \mathbb{R}^2 : \ -2 < y < 2 \ , -2 < x < 3\}$ then (2)

the given IVP has a unique solution D_3 which is the largest region.

Q1 b) ; We have $\dfrac{dy}{dx} + \left[\dfrac{1}{x} + \dfrac{1}{(x+1)\ln(x+1)}\right] y = \dfrac{2}{\ln(x+1)}$ (*)

which is a linear equation.

$$\mu(x) = e^{\int \frac{dx}{x} + \int \frac{dx}{(x+1)\ln(x+1)}} = e^{\ln x} \cdot e^{\ln \ln(x+1)} = x \ln(x+1)$$ (2)

Multiply (*) by $\mu(x)$, we get,

$$\frac{d}{dx}\left(y \times \ln(x+1)\right) = 2x$$ (2)

$$\Rightarrow \ y x \ln(x+1) = x^2 + C$$

Q2 a): $\frac{\partial M}{\partial y} = x$, $\frac{\partial N}{\partial x} = 2xy + x + 2 \rightarrow \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

We have $\frac{1}{M}\left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] = 2 \Rightarrow \mu(y) = e^{\int 2\,dy} = e^{2y}$ 1

$$\Rightarrow \underbrace{(xy e^{2y} + e^{2y})}_{M^*}dx + \underbrace{(x^2 y + \tfrac{x^2}{2} + 2x)e^{2y}}_{N^*}dy = 0$$

$$\frac{\partial M^*}{\partial y} = x e^{2y} + 2xy e^{2y} + 2e^{2y} \quad \forall \Rightarrow \frac{\partial M^*}{\partial y} = \frac{\partial N^*}{\partial x}$$

$$\frac{\partial N^*}{\partial x} = 2xy e^{2y} + x e^{2y} + 2e^{2y}$$

$\Rightarrow \exists F(x,y)$ such that $\begin{cases} \frac{\partial F}{\partial x} = xy e^{2y} + e^{2y} \longrightarrow (1) \\ \frac{\partial F}{\partial y} = x^2 y + \frac{x^2}{2} + 2x \longrightarrow (2) \end{cases}$ 1

From (1): $F(x,y) = \frac{x^2}{2} y e^{2y} + e^{2y} x + g(y) \longrightarrow (3)$

$$\frac{\partial F}{\partial y} = \frac{x^2}{2} e^{2y} + x^2 y e^{2y} + 2e^{2y} x + g'(y) \longrightarrow (4)$$ 2

(2) and (4) $\Rightarrow g'(y) = 0 \Rightarrow g(y) = C$

Hence ~~F~~ $\frac{x^2}{2} y e^{2y} + e^{2y} x = C$

Q2 b): $y' = \frac{x^2 e^x + y^3 - xy^3}{3xy^2} = \frac{y}{3}\left(\frac{1}{x} - 1\right) + \frac{x}{3} e^x y^{-2}$

$\Rightarrow y' + \frac{1}{3}\left(1 - \frac{1}{x}\right) y = \frac{x}{3} e^x y^{-2}$ (BE) 1

$\Rightarrow y' y^2 + \frac{1}{3}\left(1 - \frac{1}{x}\right) y^3 = \frac{x}{3} e^x$

Let $v = y^3 \Rightarrow 3y^2 y' = v'$

Hence $v' + \left(1 - \frac{1}{x}\right) v = x e^x$ (LE) $\longrightarrow (*)$

$\mu(x) = e^{\int (1 - \frac{1}{x})dx} = \frac{e^x}{x}$ 1

Multiply (*) by $\mu(x)$ we get

$$\frac{d}{dx}\left(\frac{e^x}{x} v\right) = e^{2x}$$ 2

$$\Rightarrow \frac{e^x}{x} y^3 = \frac{1}{2} e^{2x} + C$$

Q_3: $\underbrace{\left(x - y\tan^{-1}\left(\frac{y}{x}\right)\right)}_{M(x,y)} dx + \underbrace{x\tan^{-1}\left(\frac{y}{x}\right)}_{N(x,y)} dy = 0$

Since $M(x,y)$ and $N(x,y)$ have the same degree, then the DE is homogeneous,

$$\frac{dy}{dx} = \frac{y\tan^{-1}\left(\frac{y}{x}\right) - x}{x\tan^{-1}\left(\frac{y}{x}\right)} = \frac{\left(\frac{y}{x}\right)\tan^{-1}\left(\frac{y}{x}\right) - 1}{\tan^{-1}\left(\frac{y}{x}\right)} \quad 1$$

Let $\frac{y}{x} = u \Rightarrow y' = xu' + u$, then we have

$$xu' + u = \frac{u\tan^{-1}u - 1}{\tan^{-1}u} \Rightarrow x\frac{du}{dx} = -\frac{1}{\tan^{-1}u}$$

$$\Rightarrow \tan^{-1}u\, du = -\frac{dx}{x}$$

$$\Rightarrow \int \tan^{-1}u\, du = -\ln x + C \quad 2$$

Integrating by parts, we get

$$u\tan^{-1}u - \frac{1}{2}\ln(1+u^2) + \ln x = C$$

Then $\left(\frac{y}{x}\right)\tan^{-1}\left(\frac{y}{x}\right) - \frac{1}{2}\ln\left(1+\left(\frac{y}{x}\right)^2\right) + \ln x = C$

Simplifying, we obtain

$$\left(\frac{y}{x}\right)\tan^{-1}\left(\frac{y}{x}\right) - \frac{1}{2}\ln\left(x^2+y^2\right) + 2\ln x = C \quad 1$$

• $\underline{Q_4}$. $\dfrac{dT}{dt} = k(T - T_s)$

By integration, $T(t) = T_s + Ce^{kt}$ 1

$\left\{ \begin{array}{l} T_0 = 350°F \\ T_s = 75°F \end{array} \right.$

$T(0) = 75 + C \Rightarrow C = 350 - 75 = 275$

$T(t) = 75 + 275\, e^{kt}$

$T(15) = 150 \Rightarrow 150 = 75 + 275\, e^{15k}$ 1

$\Rightarrow e^{15k} = \dfrac{3}{11} \Rightarrow k = \dfrac{1}{15} \ln\left(\dfrac{3}{11}\right)$

Hence $T(t) = 75 + 275\, e^{\frac{t}{15}\ln\left(\frac{3}{11}\right)}$ 1

$80 = 75 + 275\, e^{\frac{t}{15}\ln\left(\frac{3}{11}\right)}$ 1

$\Rightarrow t = \dfrac{15 \ln\left(\frac{5}{275}\right)}{\ln\left(\frac{3}{11}\right)} \simeq 46.264$ minutes 1