King Saud University,
College of Sciences
Mathematical Department.

Mid-Term1 /S2/2016
Full Mark:25. Time 1H30mn
21/05/1437

Question 1[4,4]. **a)** Determine the region in the xy−plane for which the following differential equation

$$(1-y^2)\frac{dy}{dx} = xe^x,$$

would have a unique solution through the origin $(0,0)$.

b) Find the solution of the differential equation:

$$\frac{dy}{dx} - 2xy = e^x(1-2x).$$

Question 2[4,4]. **a)** Verify that the differential equation

$$\cos x dx + (1+\frac{2}{y})\sin x dy = 0, \quad y \neq 0,$$

is not exact. Find a suitable integrating factor to convert it to an exact equation, and then solve it.

b) Solve the initial value problem

$$\begin{cases} \frac{dy}{dx} = \frac{x}{y} + \frac{y}{x} \\ y(1) = 2 \end{cases} \quad x \neq 0, \quad y \neq 0$$

Question 3[4]. Find the general solution of the differential equation

$$\frac{dy}{dx} + \frac{\tan x}{2}y = \frac{(4x+5)^2}{2\cos x}y^3, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}.$$

Question 5[5]. A thermometer is removed from a room where the air temperature is 70^0F to outside where the temperature is 10^0F. After 1/2 minute the thermometer reads 50^0F. What is reading at $t = 1$ minute?. How long will it take for the thermometer to reach 15^0F.

Solutions to Math 204 Mid I(36/37)S2 (Exam held on: 21-05-1437; March 1, 2016)

Solution to Question 1

(a) $\frac{dy}{dx} = \frac{xe^x}{(1-y^2)} = f(x,y)$.
Then $\frac{\partial f}{\partial x}(x,y) = \frac{-xe^x(-2y)}{(1-y^2)^2} = \frac{2xye^x}{(1-y^2)^2}$. (1)

$\frac{dy}{dx} = f(x,y)$, $y(0) = 0$ has unique solution on the region containing $(0,0)$ whence f and $\frac{\partial f}{\partial y}$ are continuous.
f and $\frac{\partial f}{\partial y}$ are continuous on $\{(x,y)\colon y < -1\} \bigcup \{(x,y)\colon -1 < y < 1\} \bigcup \{(x,y)\colon y > 1\}$. (2)
It follows that the requested region is: $\{(x,y)\colon -1 < y < 1\}$. (1)

(b) Here $P(x) = -2x$ and $Q(x) = e^x(1-2x)$.
Integrating factor: $\psi(x) = e^{\int P(x)dx} = e^{\int -2x dx} = e^{-x^2}$. (1)

So, $\int \psi(x)Q(x)dx = \int e^{-x^2}e^x(1-2x)dx = \int (1-2x)e^{x-x^2}dx = e^{x-x^2} + C$. (1)

Hence the final solution is $y(x) = e^{x^2}\left(e^{x-x^2} + C\right)$, i.e., $y(x) = e^x + ce^{x^2}$. (2)

Solution to Question 2

(a) Here $\frac{\partial M}{\partial y} = 0$, $\frac{\partial N}{\partial x} = (1 + \frac{2}{y})\cos x \Leftrightarrow \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \Rightarrow$ the equation is not exact.

We have $\frac{(M_y - N_x)}{N} = -\cot x$. So the integrating factor: $e^{-\int \cot x dx} = \csc x$. (1)

Let $M = \cos x \csc x = \cot x$ and $N = (1 + \frac{2}{y})\sin x \csc x = 1 + \frac{2}{y}$, so that $M_y = 0 = N_x$. (1)

Now from $\frac{\partial y}{\partial x} = \cot x$, we get $f(x,y) = \ln(\sin x) + h(y)$ whence $h'(y) = 1 + \frac{2}{y}$ and $h(y) = y + \ln y^2$. (2)
Hence the solution of the given differential equation is $\ln(\sin x) + y + \ln y^2 = C$.

(b) Here $f(x,y) = \frac{x}{y} + \frac{y}{x}$. Then $f(tx,ty) = \frac{tx}{ty} + \frac{ty}{tx} = f(x,y)$ implies that f is homogeneous. (1)

Now let $u = \frac{y}{x}$, we have
$\frac{dy}{dx} = x\frac{du}{dx} + u$, $\frac{x}{y} + \frac{y}{x} = u + \frac{1}{u}$ implying that $\frac{du}{dx} = (\frac{1}{u})(\frac{1}{x})$, (2)

i.e., $udu = \frac{dx}{x}$ gives $\frac{u^2}{2} = \ln|x| + C \Rightarrow u^2 = 2\ln|x| + C$. So, $\frac{y^2}{x^2} = 2\ln|x| + C$. 1

Since $y(1) = 2$, $C = 4$, the solution is $y^2 = x^2(2\ln|x| + 4)$

Solution to Question 3

Diving both sides of the given differential equation by y^3, one obtains:

$y^{-3}\frac{dy}{dx} + (\frac{1}{2}\tan x)y^{-2} = \frac{(4x+5)^2}{2\cos x}$. Letting $u = y^{-2}$, we have 1

$\frac{du}{dx} - (\tan x)u = -\frac{(4x+5)^2}{\cos x}$. Integrating factor is: $e^{\int -\tan x dx} = e^{\int -\frac{\sin x}{\cos x}dx} = e^{\ln|\cos x|} = \cos x$. 1

Thus, we have $u = \frac{1}{\cos x}\int -\cos x\frac{(4x+5)^2}{\cos x}dx + \frac{C}{\cos x} \Rightarrow u\cos x = -\frac{1}{12}(4x+5)^3 + C$,

i.e., $\frac{1}{y^2} = -\frac{1}{12\cos x}(4x+5)^3 + \frac{C}{\cos x}$. 2

Solution to Question 4

Here $T_m = 10^oF$. So, we have the DE: $\frac{dT}{dt} = k(T-10) \Rightarrow T(t) = 10 + ce^{kt}$. 1

As $T(0) = 70$, one obtains: $70 = 10 + ce^0$ implies $c = 60^oF$ so that $T(t) = 10 + 60e^{kt}$. 1

Also, as given $T(\frac{1}{2}) = 50^oF$, we get $50 = 10 + 60e^{\frac{k}{2}} \Rightarrow k = 2\ln(\frac{4}{6})$. 1

Hence $T(t) = 10 + 60e^{2\ln(\frac{4}{6})t}$. Now at $t = 1$, we get $T(1) = 10 + 60e^{\ln(\frac{16}{36})} = 10 + 60(\frac{16}{36}) = 10 + 26.6 = 36.7^oF$. 1

If, then $T(t) = 15^o$, we get $15 = 10 + 60e^{\ln(\frac{16}{36})t} \Rightarrow t = \frac{\ln(\frac{1}{12})}{\ln(\frac{16}{36})} = 3.06$ min. 1

Q:1 (b)

$$\frac{dy}{dx} - 2xy = e^{x}(1-2x)$$

$$P = -2x \Rightarrow e^{-\int 2x\,dx} = e^{-x^2}$$

$$y\,e^{-x^2} = \int e^{-x^2} e^{x}(1-2x)\,dx + c$$

$$= \int e^{x-x^2}(1-2x)\,dx + c$$

$$= e^{x-x^2} + c$$

$$y = e^{x} + c\,e^{x^2}$$

Q.2 (a)

$$\cos x\,dx + (1+\tfrac{2}{y})\sin x\,dy = 0, \quad y \neq 0$$

$$M = \cos x, \quad N = (1+\tfrac{2}{y})\sin x$$

$$\frac{\partial M}{\partial y} = 0, \quad \frac{\partial N}{\partial x} = (1+\tfrac{2}{y})\cos x \Rightarrow \frac{\partial M}{\partial x} \neq \frac{\partial N}{\partial x}$$

$$M_y - N_x = 0 - (1+\tfrac{2}{y})\cos x$$

$$\frac{1}{N}(M_y - N_x) = -\cot x$$

$$I.F. = e^{-\int \cot x\,dx} = e^{\ln|\csc x|} = \csc x$$

$$\csc x \cos x\,dx + (1+2/y)\,dy = 0$$

$$\boxed{\cot x\,dx + (1+\tfrac{2}{y})\,dy = 0}$$

$$\frac{\partial M}{\partial y} = 0 = \frac{\partial N}{\partial x} \Rightarrow \text{exact}$$

$$\frac{\partial y}{\partial x} = \cot x \Rightarrow y = \ln|\sin x| + h(y) \Rightarrow \frac{\partial y}{\partial x} = h'(y)$$

$$h'(y) = 1 + \frac{2}{y} \Rightarrow h(y) = y + 2\ln|y| + c$$

$$\ln|\sin x| + y + \ln y^2 + c = 0$$

Q.2 (b)

$$dy/dx = \frac{x^2+y^2}{yx} = \frac{x}{y} + \frac{y}{x}$$

Put $y = ux \Rightarrow dy/dx = u + x\,du/dx$

$$u + x\,du/dx = \frac{x^2 + u^2x^2}{ux^2}$$

$$= \frac{1+u^2}{u}$$

$$x\,du/dx = \frac{1+u^2}{u} - u$$

$$= \frac{1+u^2-u^2}{u} = \frac{1}{u}$$

$$u\,du = \frac{dx}{x} \Rightarrow \frac{u^2}{2} = \ln|x| + \ln c$$

$$\frac{u^2}{2} = \ln|xc| \Rightarrow \boxed{y^2 = x^2(2\ln|x| + c)}$$

Put $x = 1, y = 2 \Rightarrow 4 = c$

$$\Rightarrow y^2 = x^2(2\ln|x| + 4)$$

Q-3 $\quad dy/dx + \frac{\tan x}{2}\, y = \frac{(4x+5)^2}{2\cos x}\, y^3$

$$y^{-3}\, dy/dx + \frac{\tan x}{2}\, y^{-2} = \frac{(4x+5)^2}{2\cos x}$$

Put $w = y^{-2}$

$$w' = dw/dx = -2y^{-3}\, dy/dx \Rightarrow -\frac{1}{2} w' = y^{-3}\, dy/dx$$

$$-\frac{1}{2} w' + \frac{\tan x}{2}\, w = \frac{(4x+5)^2}{2\cos x}$$

~~I.F~~

$$w' - 2\frac{\tan x}{2}\, w = -\frac{(4x+5)^2}{\cos x}$$

I.F $\quad e^{-\int \tan x\, dx} = \cos x$

$$w\cos x = -\int (4x+5)^2 + c$$

$$\frac{1}{y^2}\cos x = -\frac{(4x+5)^3}{4\times 3} + c$$

$$\frac{1}{y^2} = -\frac{1}{12\cos x}(4x+5)^3 + \frac{c}{\cos x}$$

Q. 4

$$\frac{dT}{dt} = k(T-10)$$

$$\frac{dT}{T-10} = k\,dt$$

$$\ln|T-10| = kt + c$$

$$T - 10 = c_1 e^{kt}$$

$$T(t) = 10 + c_1 e^{kt} \quad \text{---} \ (1)$$

$T_s = 10$

$T(0) = 70$

$T(1/2) = 50$

$$70 = 10 + c_1 \Rightarrow \boxed{c_1 = 60}$$

$$T(t) = 10 + 60e^{kt}$$

$$50 = 10 + 60e^{\frac{1}{2}k}$$

$$40 = 60e^{\frac{1}{2}k} \Longrightarrow e^{\frac{1}{2}k} = 2/3$$

$$\frac{1}{2}k = \ln(2/3)$$

$$\Rightarrow k = 2\ln(2/3)$$

$$\Rightarrow T(t) = 10 + 60e^{2\ln(2/3)t} \quad \text{---} \ (*)$$

Now after $t = 1$ minute

$$T(t) = 10 + 60e^{2(\ln 2/3)}$$

$$= 10 + 60(16/36) = 10 + 26.6$$

$$= 36.7°F$$

$\boxed{\text{If } T(t) = 15, \quad t = ?}$

By (*) $\quad T = 10 + 60e^{2\ln(2/3)t}$

$$15 = 10 + 60e^{2\ln(2/3)t} \Longrightarrow \frac{5}{60} = e^{2\ln(2/3)t}$$

$$\ln(4/9)t = \ln(1/12)$$

$$t = \frac{\ln(1/12)}{\ln(4/9)} = 3.06 \text{ min}$$