King Saud University,
College of Sciences
Mathematical Department.

Mid-Term1 /S2/2015
Full Mark:25. Time 1H30mn
14/05/1435

Question 1[4,4]. **a)** Find and sketch the largest region of the $xy-$plane for which the initial value problem

$$\begin{cases} \sqrt{9-x^2}dx + \ln(y-1)dy = 0 \\ y(1) = 4. \end{cases}$$

has a unique solution.

b) Solve the initial value problem

$$\begin{cases} \frac{dy}{dx} = xe^{-x}\sin(x) - y \\ y(0) = 1 \end{cases}$$

Question 2[4,4]. **a)** Solve the following differential equation

$$(yx\sin(x) - x^2y)dx + x(y^2 - ye^{-3y})dy = 0, \qquad y > 0,\ x > 0$$

b) Find the general solution of the differential equation

$$xy' + 2y - e^x\sqrt{y} = 0, \quad y > 0,\ x > 0.$$

Question 3[4]. Solve the differential equation

$$(y - xy)dy - (x + y^2)dx = 0, \qquad x > 1.$$

Question 4[5]. A certain culture of bacteria grows at a rate proportional to its size. let P_0 be the initial size. If the size doubles in 4 days and the size becomes $3P_0 + 750$ in 12 days. Find P_0.

P1

Answer Sheet

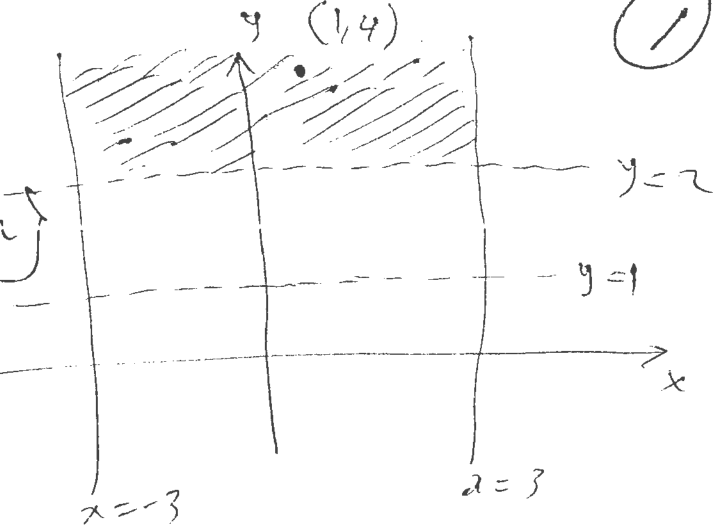
MID I Math 204 SII 2015

Q1- a) $\begin{cases} \sqrt{9-x^2}\,dx + \ln(y-1)\,dy = 0 \\ y(1) = 4 \end{cases}$

We have $y' = -\dfrac{\sqrt{9-x^2}}{\ln(y-1)} = f(x,y)$

$$\frac{\partial f}{\partial y} = + \frac{\sqrt{9-x^2}}{(y-1)\left(\ln(y-1)\right)^2}$$ 1

$f, \dfrac{\partial f}{\partial y}$ are continuous on $R_1 = \{(x,y) \in \mathbb{R}^2;\ -3 \le x \le 3,\ y > 1,\ y \ne 2\}$ 1



Since:

$(1,4) \in R_2 = \{(x,y) \in \mathbb{R}^2;\ |x| \le 3,\ y > 2\}$ 2

then the solution of the I.V.P is unique in R_2.

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b) $y' + y = x e^{-x} \sin x$ is a linear equation 1

$\mu(x) = e^{\int dx} = e^{x}$. Multiply the eq by $\mu(x)$;

Hence $\dfrac{d}{dx}\left(e^{x} y\right) = x \sin x$ 1

$\Rightarrow e^{x} y = \int x \sin x\,dx = -x\cos x + \int \cos x\,dx + C$

$= -x \cos x + \sin x + C$ 2

$\Rightarrow y = e^{-x}\left[-x\cos x + \sin x + C\right]$

P2

Q_2 - a) $\frac{dx}{x(y^2-ye^{-3y})} + \frac{dy}{yx\sin x - x^2y} = 0$, $x>0, y>0$

we have; $(yx\sin x - x^2y)dx + x(y^2-ye^{-3y})dy = 0$

which $\Rightarrow$ after simplification by xy (since $x>0$; $y>0$), we get

$(\sin x - x)dx + (y - e^{-3y})dy = 0$ (separable eq)

1

$\int (\sin x - x)\,dx = \int (e^{-3y} - y)\,dy + C$

$\Rightarrow -\cos x - \frac{x^2}{2} = -\frac{e^{-3y}}{3} - \frac{y^2}{2} + C$

2

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b) $xy' + 2y - e^{x}\sqrt{y} = 0$ This is a BE

$y' + \frac{2}{x}y = \frac{e^{x}}{x}\sqrt{y}$ $(\alpha = \frac{1}{2})$

1

Division by $\sqrt{y}$ gives

$y^{-\frac{1}{2}}y' + \frac{2}{x}y^{\frac{1}{2}} = \frac{e^{x}}{x}$ $\longrightarrow$ (*)

Let $v = y^{\frac{1}{2}} \Rightarrow v' = \frac{1}{2}y^{-\frac{1}{2}}y'$, then (*) becomes

1

$2v' + \frac{2}{x}v = \frac{e^{x}}{x}$ (Linear eq)

Standard form: $v' + \frac{v}{x} = \frac{e^{x}}{2x}$

1

$\mu(x) = e^{\int \frac{dx}{x}} = x$

Hence $\mu(x)\,v = \int \mu(x)\frac{e^{x}}{2x}\,dx = \frac{1}{2}e^{x} + C$

x

1

Q_3 $\frac{dy}{dx} = \frac{x+y^2}{y-xy}$, $x>1$ (P3)

we have $\underbrace{(x+y^2)}_{M(x,y)}dx + \underbrace{(xy-y)}_{N(x,y)}dy = 0$, $x>1$

$\frac{\partial M}{\partial y} = 2y$, $\frac{\partial N}{\partial x} = y$ (Not exact)

But $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2y-y}{y(x-1)} = \frac{1}{x-1}$

$\Rightarrow \mu(x) = e^{\int \frac{dx}{x-1}} = x-1$ (1)

Then $(x-1)(x+y^2)dx + (x-1)(xy-y)dy = 0$

is exact

$\Rightarrow \exists$ a function $F(x,y)$ such that (1)

$$\begin{cases} \frac{\partial F}{\partial x} = (x-1)(x+y^2) \longrightarrow (1) \\ \frac{\partial F}{\partial y} = (x-1)(xy-y) \longrightarrow (2) \end{cases}$$ (1)

From (2) $F(x,y) = (x-1)^2 \frac{y^2}{2} + h(x) \longrightarrow (3)$

$\frac{\partial F}{\partial x} = 2(x-1)\frac{y^2}{2} + h'(x) \longrightarrow (4)$

From (1) and (4), we have

$(x-1)(x+y^2) = (x-1)y^2 + h'(x)$

$\Rightarrow h'(x) = x^2 - x \Rightarrow h(x) = \frac{x^3}{3} - \frac{x^2}{2} + c$

Hence $(x-1)^2 \frac{y^2}{2} + \frac{x^3}{3} - \frac{x^2}{2} = C$ (1)

(P4)

Q4 - $\frac{dP}{dt} = kP \Rightarrow \frac{dP}{P} = k\,dt \Rightarrow P(t) = Ce^{kt}$

$P(0) = P_0 \Rightarrow P(t) = P_0 e^{kt}$ (1)

$P(4) = 2P_0 \Rightarrow 2P_0 = P_0 e^{4k} \Rightarrow k = \frac{\ln 2}{4}$ (1)

Consequently $P(t) = P_0 e^{\frac{t}{4}\ln 2}$.

$P(12) = 3P_0 + 750 \Rightarrow P_0 e^{3\ln 2} = 3P_0 + 750$ (1)

$\Rightarrow 8P_0 = 3P_0 + 750$ (2)

$\Rightarrow 5P_0 = 750 \Rightarrow \underline{P_0 = 150}$