King Saud University,
College of Sciences
Mathematical Department.

Mid-Term Exam /S2/2022
Full Mark:30. Time 2H
23/03/2022

Question 1. [5, 4] **a)** A radioactive substance has a half-life of 4000 years. If 200 grams were initially present, how much of the substance will be left after 10000 years.

b) Find the general solution of the differential equation

$$dy + \frac{y(x+y)}{x^2}dx = 0, \quad x > 0.$$

Question 2. [5] Find and sketch the largest region of the $xy-$plane for which the initial value problem

$$\begin{cases} \cos^{-1}(1+y)dx + (\ln(x+1) - 1)\,dy = 0 \\ y(0) = -1, \end{cases}$$

has a unique solution.

(Hint: $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$)

Question 3. [4, 4]. **a)** Solve the differential equation

$$y\frac{dy}{dx}e^{y-x} - \ln(1+e^x) = 0.$$

b) By using an appropriate integrating factor, find the general solution of the differential equation

$$\cos x dx + (2 + \frac{3}{y}) \sin x dy = 0, \quad 0 < x < \pi, \; y > 0.$$

Question 4. [4, 4]. **a)** Solve the initial value problem

$$\begin{cases} y(y-1)\sin x dx - dy = 0 \\ y(\frac{\pi}{2}) = 1 \end{cases}$$

b) Determine whether the following functions

$$f_1(x) = \ln(x+2), \; f_2(x) = \ln(2-x), \; f_3(x) = \ln(4-x^2),$$

are linearly dependent or linearly independent on the interval $(-2, 2)$.

Q_1 a) : $Q(t) = 200e^{kt}$ (1)

$Q(4000) = 100 = 200e^{4000k}$

$\Rightarrow k = \frac{\ln(\frac{1}{2})}{4000} = -\frac{\ln 2}{4000}$ (2)

$Q(10000) = 200e^{-\frac{\ln 2}{4000} \cdot 10000} = 200e^{-\frac{5\ln 2}{2}} \simeq 35.355$ gr.

(2)

Q_1 b) $\frac{dy}{dx} = -\left(\frac{y}{x}\right) - \left(\frac{y}{x}\right)^2$ (HE) (P1)

Let $u = \frac{y}{x} \Rightarrow y = xu \Rightarrow y' = u + xu'$, then (1)

$u + x\frac{du}{dx} = -u - u^2 \Rightarrow \frac{du}{u^2 + 2u} = -\frac{dx}{x}$

$\Rightarrow \int \frac{du}{u(u+2)} = -\ln x + C_1 \quad (x>0)$

By decomposition we have $\frac{1}{2}\int \frac{du}{u} - \frac{1}{2}\int \frac{du}{u+2} = -\ln x + C_1$ (1)

$\frac{1}{2}\ln|u| - \frac{1}{2}\ln|u+2| + \ln x = C_1$

$\Rightarrow \ln\left|\frac{x^2 u}{u+2}\right| = C_1 \Rightarrow \frac{x^2 u}{u+2} = C_2$ (1)

$\Rightarrow \frac{x^2\left(\frac{y}{x}\right)}{\frac{y}{x} + 2} = C_2 \Rightarrow \frac{x^2 y}{y + 2x} = C_2.$ (1)

$\underline{Q_2}$: $y' = f(x,y) = \dfrac{\cos^{-1}(y+1)}{1-\ln(1+x)}$ is continuous on (P2)

$R_1 = \{(x,y) \in \mathbb{R}^2;\ -1 \le y+1 \le 1,\ 1+x > 0,\ x \ne e-1\}$ (1)

$= \{(x,y) \in \mathbb{R}^2:\ -2 \le y \le 0,\ x \in (-1, e-1) \cup (e-1, \infty)\}$

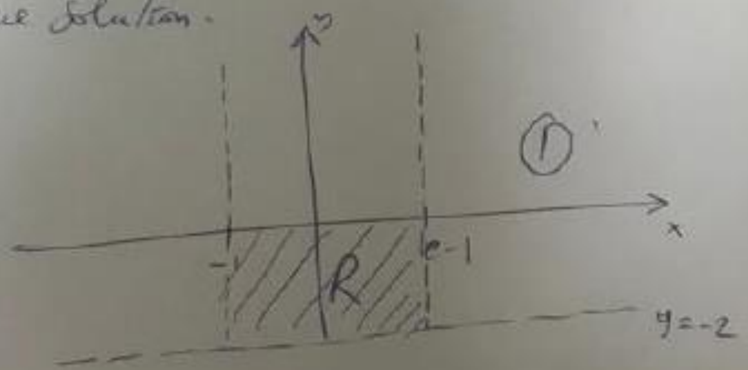
$\dfrac{\partial f}{\partial y} = \dfrac{-1}{1+\ln(1+x)} \cdot \dfrac{1}{\sqrt{1-(y+1)^2}}$ is cont on

$R_2 = \{(x,y) \in \mathbb{R}^2;\ -2 < y < 0,\ x \in (-1, e-1) \cup (e-1, \infty)\}$ (1)

$f, \dfrac{\partial f}{\partial y}$ are cont on $R_1 \cap R_2 = R_2$

or $(0,-1) \in R_* = \{(x,y) \in \mathbb{R}^2:\ -1 < x < e-1,\ -2 < y < 0\}$ (2)

which is the largest region for which the I.V.P admits a unique solution.



Q_3 a) $y\frac{dy}{dx}e^{y}\cdot e^{-x} = \ln(1+e^{x})$ (Sep Eq)

P_3

$$ye^{y}dy = e^{x}\ln(1+e^{x})\,dx$$

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$$\Rightarrow \int y e^{y}dy = \int e^{x}\ln(1+e^{x})\,dx$$

$1+e^{x}=t$

$\Rightarrow e^{x}dx = dt$

$$\Rightarrow \quad ye^{y}-e^{y} = \int \ln t\,dt$$

$$= t\ln t - t + C$$

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$$= (1+e^{x})\ln(1+e^{x}) - (1+e^{x}) + C$$

Q_3 b) : $\underbrace{\cos x}_{M}\, dx + \underbrace{\left(2+\frac{3}{y}\right)\sin x}_{N}\, dy = 0 \longrightarrow (*)$ P4

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{-\left(2+\frac{3}{y}\right)\cos x}{\left(2+\frac{3}{y}\right)\sin x} = -\cot x$$

Thus $\mu(x) = e^{-\int \cot x\, dx} = \frac{1}{\sin x}$ (1)

Multiply (*) by $\mu(x) = \frac{1}{\sin x}$

$$\frac{\cos x}{\sin x}\, dx + \left(2+\frac{3}{y}\right) dy = 0 \rightarrow (**)$$

$$\frac{\partial M}{\partial y} = 0, \quad \frac{\partial N}{\partial x} = 0$$

Eq (**) is exact $\Rightarrow \exists\, F(x,y)$ such that

$$\begin{cases} \frac{\partial F}{\partial x} = \frac{\cos x}{\sin x} \longrightarrow (1) \\ \frac{\partial F}{\partial y} = \frac{3}{y} + 2 \longrightarrow (2) \end{cases}$$ (1)

From (2), $F(x,y) = 3\ln y + 2y + B(x) \longrightarrow (3)$

$$\frac{\partial F}{\partial x} = B'(x) \longrightarrow (4)$$

From (1) and (4), we have $B'(x) = \frac{\cos x}{\sin x}$

$$\Rightarrow B(x) = \int \frac{\cos x}{\sin x}\, dx = \ln|\sin x| + C_1$$

$$= \ln(\sin x) + C_1 \quad (0<x<\pi)$$ (2)

Hence $3\ln y + 2y + \ln(\sin x) = C$

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P5

Q4 a) $\begin{cases} y(y-1)\sin x\, dx - dy = 0 \\ y(\frac{\pi}{2}) = 1 \end{cases}$

The DE can be treated as separable eq or Bernoulli equation

$$\int \sin x\, dx = \int \frac{dy}{y(y-1)}$$ 1

$$\frac{1}{y(y-1)} = \frac{A}{y} + \frac{B}{y-1} = \frac{y(A+B) - A}{y(y-1)} \Rightarrow A = -1,\ B = 1$$

Hence $-\cos x = -\ln|y| + \ln|y-1| + C_1$

$(C_2 = -C_1)$

$$\Rightarrow \ln\left|\frac{y-1}{y}\right| = -\cos x + C_2$$ 2

$$\Rightarrow \left|\frac{y-1}{y}\right| = e^{-\cos x} \cdot e^{C_2}$$ $(C_3 = \pm e^{C_2})$

$$\Rightarrow \frac{y-1}{y} = C_3 e^{-\cos x}$$

$$-\frac{1}{y} = C_3 e^{-\cos x} - 1$$

$y(\frac{\pi}{2}) = 1 \Rightarrow -1 = C_3 - 1 \Rightarrow C_3 = 0$ 1

Hence $y = 1$ (Singular Solution)

Another solution:

$$(y^2 \sin x - y \sin x)dx = dy$$

$$\frac{dy}{dx} = y^2 \sin x - y \sin x$$

$$y' + y \sin x = y^2 \sin x$$

$$y^{-2} y' + y^{-1} \sin x = \sin x$$

Let $u = y^{-1} \Rightarrow u' = -y^{-2} y'$

Then $-u' + u \sin x = \sin x$

$$\Rightarrow \quad u' - u \sin x = -\sin x$$

$$\mu(x) = e^{-\int \sin x \, dx} = e^{\cos x}, \text{ then}$$

$$\frac{d}{dx}\left(e^{\cos x} u\right) = -\sin x \, e^{\cos x}$$

$$\Rightarrow \quad e^{\cos x} u = -\int \sin x \, e^{\cos x} dx = e^{\cos x} + C$$

$$\frac{1}{y} = u = 1 + C e^{-\cos x}$$

$$\Rightarrow \quad y = \frac{1}{1 + C e^{-\cos x}}$$

$$y\left(\frac{\pi}{2}\right) = 1 = \frac{1}{1+C} \Rightarrow C = 0$$

Then $y = 1$

6) $f_3(x) = \ln(4-x^2) = \ln(2-x)(2+x) = \ln(2-x) + \ln(2+x)$

that is $f_3(x) - f_2(x) - f_1(x) = 0$

Hence f_1, f_2, f_3 are linearly dependent on $(-2, 2)$

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