King Saud University, Department of Mathematics
Math 204 (2H), 30/100, Mid term Exam S2. 41/42

Question 1[5,4] **a)** Find and sketch the largest region in $\mathbb{R}^2$, for wich the following initial value problem admits a unique solution

$$\begin{cases} (2^{\sqrt{y}}+\sqrt{x+y})\,dx-\ln(1-x^2)dy=0 \\ y(-\frac{1}{2})=1. \end{cases}$$

b) Solve the differential equation

$$\tan y - x\frac{dy}{dx}=4x^2\tan y,\quad y\in(0,\frac{\pi}{2}),\ x>0.$$

Question 2. [4,4]. **a)** Solve the initial value problem

$$\begin{cases} \frac{dy}{dx}=3-\sqrt{x+y-1} \\ y(0)=2. \end{cases}$$

b) Find the general solution of the differential equation

$$\left(x\cos\frac{y}{x}+y\right)dx-xdy=0,\quad x\neq 0.$$

Question 3. [4,4] **a)** Use the substitution $u=\ln y$ to reduce the differential equation

$$x\frac{dy}{dx}=2x^2y+y\ln y,\quad x>0,\ y>0.$$

to a linear equation, and then solve it.

b) Obtain the general solution of the following differential equation

$$(ye^{-2x}+y^3)dx-e^{-2x}dy=0,\quad y\neq 0.$$

Question 4 [5] Initially there were 60 grams of a radioactive material present. After 8 hours the mass decreases by 4%. We suppose that the rate of decay is proportional to the amount of the material at time t. Determine the half life of this material.

Q. a) ~~Find~~ Find and sketch the largest region in $\mathbb{R}^2$ on which the following I.V.P has a unique solution

Question ① ⓐ

$$\begin{cases} \left(2^{\sqrt{y}} + \sqrt{x+y}\right)dx - \ln(1-x^2)\,dy = 0 \\ y\left(-\frac{1}{2}\right) = 1 \end{cases}$$

Solution.

$\left(2^{\sqrt{y}} + \sqrt{x+y}\right)dx - \ln(1-x^2)\,dy = 0$

$$\Rightarrow \frac{dy}{dx} = \boxed{\frac{2^{\sqrt{y}} + \sqrt{x+y}}{\ln(1-x^2)} = f(x,y).}$$

- $\dfrac{\partial f}{\partial y}(x,y) = \left(\dfrac{2^{\sqrt{y}}\cdot \ln 2}{2\sqrt{y}} + \dfrac{1}{2\sqrt{x+y}}\right)\cdot\dfrac{1}{\ln(1-x^2)}$ 1

f is continuous on $\{(x,y) : 1-x^2 > 0$ and $1-x^2 \neq 1$; $y \geq 0$ and $x+y \geq 0\}$.

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$\dfrac{\partial f}{\partial y}$ is continuous on $\{(x,y) : 1-x^2 > 0$ and $1-x^2 \neq 1$; $y > 0$ and $x+y > 0\}$.

ince ;

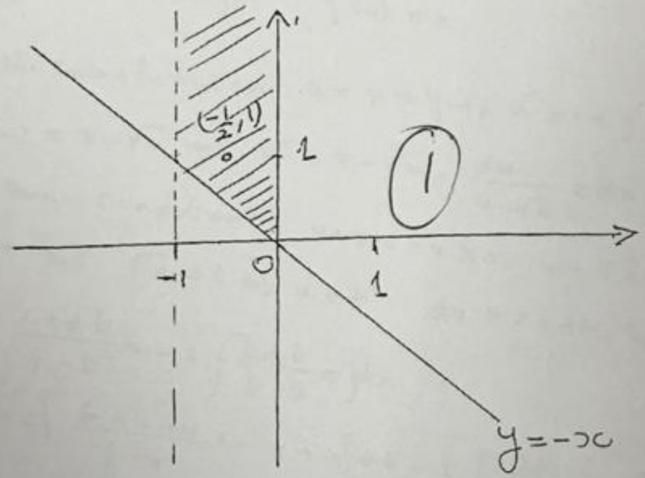
f and $\frac{\partial f}{\partial y}$ are Continuous

on. $\{(x,y): \ x \in (-1,0) \cup (0,1).$
$y > 0$ and $y + x > 0$ ②

Since $y(-\frac{1}{2}) = 1$; the

requested region. is:

$\{(x,y): \ x \in (-1,0) \ ; y > -x\}$



Question 1 (b)

$$\tan y - x\frac{dy}{dx} = 4x^2 \tan y \qquad y \in (0, \pi/2),\ x > 0$$

$$(1-4x^2)\tan y = x\frac{dy}{dx}$$

$$\frac{dy}{\tan y} = \left(\frac{1}{x} - 4x\right)dx \qquad 2$$

$$\int \frac{\cos y}{\sin y}\,dy = \int \left(\frac{1}{x} - 4x\right)dx$$

$$\ln(\sin y) = \ln x - 2x^2 + c_1 \Rightarrow \boxed{\sin y = c\, x\, e^{-2x^2}} \qquad 2$$

where $c = e^{c_1}$.

Question 2

(a) $\dfrac{dy}{dx} = 3 - \sqrt{x+y-1}$, $y(0) = 2$

We put the substitution $u = x+y-1$, $u' = 1 + y' \Rightarrow y' = u' - 1$.

Then $u' - 1 = 3 - \sqrt{u} \Rightarrow u' = 4 - \sqrt{u}$, $\dfrac{du}{4-\sqrt{u}} = dx$.

There is some conditions: $x+y-1 = u \geq 0$, $4 - \sqrt{u} \neq 0$

Now, we put $\sqrt{u} = t \Rightarrow u = t^2$, $du = 2t\,dt$, hence 1

$$\int \frac{2t\,dt}{4-t} = -2\int \frac{t\,dt}{t-4} = \int dx$$

$$-2\int \frac{t-4+4}{t-4}\,dt = -2\int dt - 8\int \frac{dt}{t-4} = \int dx$$

$$-2t - 8\ln|t-4| = x + c \qquad 1$$

or $\quad -2\sqrt{u} - 8\ln|\sqrt{u} - 4| = x + c$

$$\boxed{-2\sqrt{x+y-1} - 8\ln\left|\sqrt{x+y-1} - 4\right| = x + c}$$

From $y(0) = 2 \Rightarrow -2 - 8\ln|1-4| = 0 + c$

$$\boxed{c = -2 - 8\ln 3} \qquad 1$$

Then the solution of the IVP is

$$\boxed{-2\sqrt{x+y-1} - 8\ln\left|\sqrt{x+y-1} - 4\right| = x - 2 - 8\ln 3} \qquad 1$$

(b) $[x\cos(\frac{y}{x}) + y]dx - x\,dy = 0$; $x \neq 0$

Let $u = \frac{y}{x} \Rightarrow y = xu$, $dy = x\,du + u\,dx$

$[\cos(\frac{y}{x}) + \frac{y}{x}]dx - dy = 0$ (1)

$(\cos u + u)dx - (u\,dx + x\,du) = 0$

$\cos u\,dx + x\,du = 0 \Rightarrow \int \frac{dx}{x} + \int \sec u\,du =$ (1)

$\ln|x| + \ln|\sec u + \tan u| = c_1$

with some condition $0 < u = \frac{y}{x} < \frac{\pi}{2}$

Then $\ln|x| + \ln|\sec(\frac{y}{x}) + \tan(\frac{y}{x})| = c_1$

or $\boxed{x(\sec(\frac{y}{x}) + \tan(\frac{y}{x})) = c}$, $(c = \mp e^{c_1})$ (2)

Question (3)

(a) $x\frac{dy}{dx} = 2x^2y + y\ln y$; $y > 0$, $x > 0$

We put $u = \ln y \Rightarrow u' = \frac{y'}{y}$

$x\frac{y'}{y} = 2x^2 + \ln y \Rightarrow xu' = 2x^2 + u$ (1)

then $u' - \frac{1}{x}u = 2x$ is linear D.E.

$\mu(x) = e^{\int -\frac{1}{x}dx} = e^{-\ln x} = \frac{1}{x}$ (1)

$\mu(x)u = \int \mu(x)(2x)$

$\frac{u}{x} = \int 2\,dx = 2x + c$ or $u = 2x^2 + cx$

Then the solution of the D.E is $\ln y = 2x^2 + cx$ (2)

or $\boxed{y = e^{2x^2 + cx}}$

(b) $(ye^{-2x}+y^3)dx - e^{-2x}dy = 0$

We have $y' = \frac{dy}{dx} = \frac{ye^{-2x}+y^3}{e^{-2x}} = y + e^{2x}y^3$

Or $y' - y = e^{2x}y^3$ is B. E. with $n=3$ (1)

We suppose that $y \neq 0$ on some interval I, then

$y'y^{-3} - y^{-2} = e^{2x}$; we put $u = y^{-2} \Rightarrow u' = -2y^{-3}y'$

So $-\frac{u'}{2} - u = e^{2x} \Rightarrow u' + 2u = -2e^{2x}$ which is linear, $\mu(x) = e^{\int 2dx} = e^{2x}$ (1)

$\mu(x)u = ue^{2x} = -2\int e^{2x}.e^{2x}dx$

$ue^{2x} = -2\int e^{4x}dx = -\frac{1}{2}e^{4x} + c$

So the solution of the D.E is

$y^{-2}.e^{2x} = -\frac{1}{2}e^{4x} + c$ (2)

$$\boxed{y^{-2} = -\frac{1}{2}e^{2x} + ce^{-2x}}$$

Question (4) $\frac{dm(t)}{m(t)} = kt \Rightarrow \ln(m(t)) = kt + c$

$m(t) = e^c.e^{kt} = c_1e^{kt}$; $c_1 = e^c$

$m(0) = 60 \Rightarrow m(0) = c_1 = 60 \Rightarrow m(t) = 60e^{kt}$ (1)

$m(8) = 60 - 60(\frac{4}{100}) = 60 - \frac{24}{10} = 60 - 2.4 = 57.6$

$m(8) = 60e^{k(8)} = 57.6$

$0.96 = \frac{57.6}{60} = e^{8k} \Rightarrow 8k = \ln(0.96) \Rightarrow \boxed{k = \frac{1}{8}\ln(0.96)}$ (2)

$m(t) = 60\, e^{\frac{1}{8}\ln(0.96)t}$

Now we have to find t s.t $m(t) = 30$

$$30 = 60\, e^{\frac{1}{8}\ln(0.96)t}$$

$$\frac{1}{2} = e^{\left(\ln(0.96)^{1/8}\right)t}$$

$$-\ln 2 = t\ln(0.96)^{1/8} = t\cdot\frac{1}{8}\ln(0.96)$$

$$-8\ln 2 = t\ln(0.96) \Rightarrow t = \frac{-8\ln 2}{\ln(0.96)} = \frac{-5.545}{-0.041}$$

$$\boxed{t \neq 135.244 \text{ hours.}}$$

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