King Saud University,
College of Sciences
Mathematical Department.

Mid-Term1 /S2/2020
Full Mark:25. Time 1H30mn
5/3/2020

Question 1[4]. Find and sketch the largest local region of the xy−plane for which the initial value problem

$$\begin{cases} \ln(4-x^2)\frac{dy}{dx} = \sqrt{1-x} + y\sin^{-1} x \\ y(\frac{1}{2}) = 0, \end{cases}$$

has a unique solution.

Question 2[4+4]. **a)** Solve the initial value problem

$$\begin{cases} xy^2dy - (x^3+2y^3)dx = 0, \quad y \neq 0,\ x \neq 0, \\ y(1) = 1. \end{cases}$$

b) Reduce the following differential equation to a Bernoulli equation and then obtain its general solution

$$y(6y^2 - 1 - x)dx + 2xdy = 0; \quad x > 0.$$

Question 3[4+4]. **a)** Test wether the following differential equation is exact or not, and then solve it

$$\left(\frac{(x+y)^3}{3x} + (x+y)^2\right)dx + (x+y)^2dy = 0, \quad x > 0,\ x+y \neq 0.$$

b) Find the general solution of the differential equation

$$\begin{cases} \frac{dy}{dx} = e^{y-x}(1+x)\sec y, \quad -\frac{\pi}{2} < y < \frac{\pi}{2} \\ y(0) = 0. \end{cases}$$

Question 4[5]. A new car is worth 80000 rials. If the price decreases 15% ~~each year,~~ what will be worth in 6 years.

~~So~~ after one year

First Mid-term exam M. 204

With Complete solution 1441 H

Question 1

$$y' = f(x,y) = \frac{\sqrt{1-x} + (\sin^{-1}x)\,y}{\ln(4-x^2)}$$

$$\frac{\partial f}{\partial y} = \frac{\sin^{-1}(x)}{\ln(4-x^2)}.$$

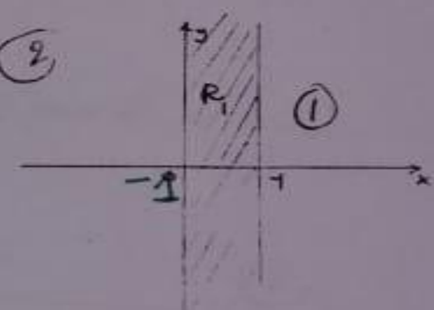
Then f and $\frac{\partial f}{\partial y}$ are continuous on:

$$R = \left\{ (x,y) ;\ \begin{array}{l} -1 \le x \le 1,\ 4-x^2 > 0,\ \text{s.t. } 4-x^2 \neq 1 \\ \text{and } 1-x \ge 0,\ y \in \mathbb{R} \end{array} \right\}.$$

$$= \{ (x,y) ;\ -1 \le x \le 1,\ y \in \mathbb{R} \} \quad (1)$$

But $(\frac{1}{2}, 0) \in R$, then the largest local region in xy-plane for which the IVP has a unique solution is

$$R_1 = \{ (x,y),\ -1 \le x \le 1,\ \cancel{x}\ \ y \in \mathbb{R} \} \quad (2)$$



Question 2

(a) $\begin{cases} xy^2\,dy - (x^3+2y^3)\,dx = 0 ; & y \neq 0,\ x \neq 0 \\ y(1) = 1 \end{cases}$

The D.E is homogeneous, so we put $\frac{y}{x} = u \Rightarrow y = xu$ and $dy = u\,dx + x\,du$

$$\left(\frac{y}{x}\right)^2 dy - \left[1 + 2\left(\frac{y}{x}\right)^3\right] dx = 0 \quad (1)$$

$$u^2(x\,du + u\,dx) - (1+2u^3)\,dx = 0, \quad x u^2\,du - (1+u^3)\,dx = 0$$

$$\frac{dx}{x} - \frac{u^2\,du}{1+u^3} = 0 \Rightarrow \ln|x| - \frac{1}{3}\ln|1+u^3| = c$$

$$\ln|x|^3 - \ln|1+u^3| = c \quad (2)$$

$$\ln\left|\frac{x^3}{1+u^3}\right| = c \Rightarrow \ln\left|\frac{x^6}{x^3+y^3}\right| = c \quad \text{or} \quad \boxed{x^6 = (x^3+y^3)\,c_1}$$

where $c_1 = \mp e^c$

But $y(1)=1$, then $1 = 2c_1 \Rightarrow c_1 = \frac{1}{2}$ (1)

So the unique solution of the IVP is $\boxed{x^3+y^3-2x^6=0}$

ⓑ $y(6y^2-1-x)dx + 2x\,dy = 0$; $x>0$

$6y^3-(1+x)y + 2xy' = 0$ which is a Bernoulli's D.E

$y' - \frac{(1+x)}{2x} y = -\frac{3}{x} y^3$, $n=3$ (1)

We suppose that $y \neq 0$ on some interval I. Then we have

$y' y^{-3} - \frac{1+x}{2x} y^{-2} = -\frac{3}{x}$,

Let $y^{-2} = u \Rightarrow u' = -2y^{-3}y'$, $\frac{-u'}{2} = y^{-3}y'$

$\frac{-u'}{2} - \frac{1+x}{2x} u = -\frac{3}{x} \Rightarrow u' + (1+\frac{1}{x})u = \frac{6}{x}$ is linear

$\mu(x) = e^{\int(1+\frac{1}{x})dx} = xe^x$ (1)

$u\mu(x) = xe^x u = \int \frac{6}{x} xe^x dx = 6e^x + c$, hence

$\boxed{y^2(6e^x+c) = xe^x}$ or $\boxed{y^2(6+ce^{-x}) = x}$ (2)

Question 3

ⓐ $\underbrace{\left[\frac{(x+y)^3}{3x} + (x+y)^2 \right]}_{M} dx + \underbrace{(x+y)^2}_{N} dy = 0$, $x>0$ and $y+x \neq 0$

$\frac{\partial M}{\partial y} = \frac{(x+y)^2}{x} + 2(x+y)$, $\frac{\partial N}{\partial x} = 2(x+y)$, So the D.E. is not exact (1)

But $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{1}{x} = f(x)$, $\Rightarrow \mu(x) = e^{\int \frac{dx}{x}} = x$ is an integrating factor. Then

$\underbrace{\left[\frac{(x+y)^3}{3} + x(x+y)^2\right]}_{M} dx + \overbrace{x(y+x)^2}^{N} dy = 0$ is exact D.E, because

$\frac{\partial M}{\partial y} = (x+y)^2 + 2x(x+y) = \frac{\partial N}{\partial x}$ (1)

Then there is a function F of x and y s.t

$$\frac{\partial F}{\partial x} = M = \frac{1}{3}(x+y)^3 + x(x+y)^2$$

$$\frac{\partial F}{\partial y} = N = x(y+x)^2$$

So $F(x,y) = \int x(y+x)^2 dy = x\,\frac{(x+y)^3}{3} + \phi(x)$ (2)

$$\frac{\partial F}{\partial x} = \frac{1}{3}(x+y)^3 + x(x+y)^2 + \phi'(x) = \frac{1}{3}(x+y)^3 + x(x+y)^2$$

$$\Rightarrow \phi'(x) = 0 \Rightarrow \phi(x) = c$$

So The solution of the D.E is

$$F(x,y) = x\,\frac{(x+y)^3}{3} + c = 0 \quad \text{or} \quad \boxed{x(x+y)^3 + c_1 = 0} \quad (c_1 = 3c)$$

(b) $\begin{cases} \frac{dy}{dx} = e^{y-x}(1+x)\sec y \\ y(0) = 0 \end{cases}$

$$dy = e^{y}.e^{-x}(1+x)\sec y\, dx$$

$$\int e^{-y}\cos y\, dy = \int e^{-x}(1+x)\,dx \Rightarrow \boxed{\frac{e^{-y}}{2}(\sin y - \cos y) = -e^{-x}(x+2) + c}$$ (2)

By using the initial condition $y(0)=0 \Rightarrow \frac{1}{2}(0-1) = -(0+2) + c$

$c = \frac{3}{2}$. Hence the solution of the IVP is (2)

$$\boxed{\frac{e^{-y}}{2}(\sin y - \cos y) = \frac{3}{2} - e^{-x}(x+2)} \quad \text{or}$$

$$\boxed{e^{-y}(\sin y - \cos y) + 2e^{-x}(x+2) = 3}$$

Question 4 $\frac{dP}{P} = k\,dt \Rightarrow \ln P = kt + c \Rightarrow P(t) = c_1 e^{kt}$ (1)

$P(0) = 80000 \Rightarrow P(t) = 80000e^{kt}$

$P(1) = 80000 - (80000)(\frac{15}{100}) = 80000 - 12000 = \boxed{68000 = 80000e^{k}}$ (1)

$$k = \ln\left(\frac{68}{80}\right) = \ln\left(\frac{17}{20}\right) = \ln(0.85) \simeq -0.163 \quad (1)$$

$\boxed{P(t) \cong 80000e^{-0.163t}}$, So $P(6) \cong 80000\,e^{(-0.163)6} = 80000e^{-0.978}$

$\boxed{P(6) \simeq 30085}$ Rials. (2)