King Saud University,
College of Sciences
Mathematical Department.

Mid-Term1 /S1/2018
Full Mark:25. Time 1H30mn
18/10/2018

Question 1[5]. Find and sketch the largest local region of the $xy-$plane for which the initial value problem

$$\begin{cases} \sqrt{y^2-4}dy - (x-y)^2 \ln x dx = 0 \\ y(2) = -3, \end{cases}$$

has a unique solution.

Question 2[4+4]. **a)** Solve the initial value problem

$$\begin{cases} \frac{dy}{dx} = \frac{y}{x}(\ln x - \ln y), \quad x > 0, \ y > 0. \\ y(1) = 1. \end{cases}$$

b) By using an appropriate substitution, solve the differential equation

$$2xe^{2y}\frac{dy}{dx} = 3x^4 + e^{2y}, \quad x > 0$$

Question 3[4+4]. **a)** Solve the differential equation

$$2y + 2x^2y^2 + (x + x^3y)\frac{dy}{dx} = 0, \quad x + x^3y \neq 0.$$

b) Find the general solution of the differential equation

$$(2x^{-1}y^{3/2} - x^{-2}e^x)dx + \sqrt{y}dy = 0, \quad x > 0, \quad y > 0.$$

Question 4[4]. The population of a town is doubled in 5 years and became 20000 in 10 years. What is the initial population if the rate of growth of population is proportional to the population at that instant.

Complete solutions of M.204
Mid - Term 1. S.1.2018

Question ①

$$\sqrt{y^2-4}\,dy = (x-y)^2 \ln x\,dx$$

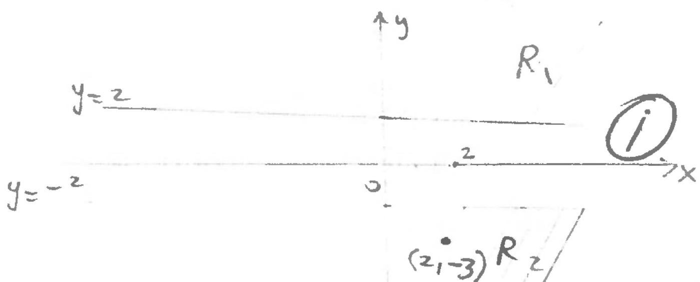
$$y' = \frac{(x-y)^2 \ln x}{\sqrt{y^2-4}} \quad ; \quad x>0 \text{ and } y^2>4$$

$$y' = f(x,y)$$

f is continuous on $R = \{(x,y) ; x>0 , |y|>2\}$ 1

$$\frac{\partial f}{\partial y} = \frac{-2(x-y)}{\sqrt{y^2-4}}\ln x - \frac{y(x-y)^2}{(y^2-4)^{3/2}}\ln x$$ 1

$\frac{\partial f}{\partial y}$ is also continuous on $R = \underbrace{\{(x,y) ; x>0, y>2\}}_{R_1} \cup \underbrace{\{(x,y) ; x>0, y<-2\}}_{R_2}$ 1



But as $(2,-3) \in R_2$

Then the longest local region of the xy-plane for which the IVP has a unique solution is R_2 1

Question ②

ⓐ $y' = \frac{y}{x}(\ln x - \ln y) = \frac{y}{x}\ln\left(\frac{x}{y}\right) = -\frac{y}{x}\ln\left(\frac{y}{x}\right),$

$y(1) = 1$

We put $u = \frac{y}{x} \Rightarrow y = xu, \quad y' = u + xu'$, then

$$u + xu' = -u\ln u$$

$$xu' + u(1+\ln u) = 0$$

$$x\,du + u(1+\ln u)\,dx = 0$$

$$\int \frac{du}{u(1+\ln u)} + \int \frac{dx}{x} = 0$$ 1

$$\ln|\ln u + 1| + \ln x = c$$

$$\ln|x(\ln u + 1)| = c$$ 1

$$x\,|\ln u + 1| = e^c$$

$$x(\ln u + 1) = \mp e^c = c_1 \neq 0$$

$$\boxed{x(1 + \ln(y/x)) = c_1}$$

But $y(1) = 1$, then $1(1 + \ln 1) = 1 = c_1$ (1)

So the solution of the IVP is $\boxed{x(1 + \ln(\frac{y}{x})) = 1}$ (1)

(b)

$$2x\, e^{2y}\, y' = 3x^4 + e^{2y},$$

We put (1) $u = e^{2y}$, $u' = 2y' e^{2y}$, then we have

$$x u' = 3x^4 + u \quad \text{or}$$

$$u' - \frac{u}{x} = 3x^3 \quad (\text{is linear D.E})$$

$P(x) = -\frac{1}{x}$, $\mu(x) = e^{\int -\frac{dx}{x}} = \frac{1}{x}$ (1)

Hence $u(\frac{1}{x}) = \int 3x^3(\frac{1}{x})dx = \int 3x^2 dx = x^3 + c$ (1)

Then the solution of the D.E is given by

$$\boxed{e^{2y} = x(x^3 + c)} \quad \text{or} \quad \boxed{y = \frac{1}{2}\ln(cx + x^4)} \quad (1)$$

Question (3)

(a) $(\overbrace{2y + 2x^2y^2}^{M})dx + (\overbrace{x + x^3y}^{N})dy = 0$; $x + x^3y \neq 0$

$\frac{\partial M}{\partial y} = 2 + 4x^2y$, $\frac{\partial N}{\partial x} = 1 + 3x^2y$, then $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ (1)

So the D.E. is not exact.

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{1 + x^2y}{x(1 + x^2y)} = \frac{1}{x} = f(x), \quad \mu(x) = e^{\int \frac{dx}{x}} = |x| \quad (1)$$

We can take $\mu(x) = x$ or $\mu(x) = -x$.

Now the D.E :

$$(2xy + 2x^3y^2)dx + (x^2 + x^4y)dy = 0$$

becomes exact because

$$\frac{\partial M}{\partial y} = 2x + 4x^3y = \frac{\partial N}{\partial x}$$

Then $\exists$ F of x and y s.t

$$\frac{\partial F}{\partial x} = 2xy + 2x^3y^2, \quad \frac{\partial F}{\partial y} = x^2 + x^4 y$$

$$F(x,y) = \int (2xy + 2x^3y^2)\,dx = x^2y + \frac{1}{2}x^4y^2 + \phi(y) \quad 1$$

$$\frac{\partial F}{\partial y} = x^2 + x^4 y + \phi'(y) = x^2 + x^4 y \Rightarrow \phi'(y) = 0$$

hence $\phi(y) = c$ and the Solution of the D.E. is

$$\boxed{F(x,y) = x^2y + \frac{1}{2}x^4y^2 + c = 0} \quad 1$$

(b) $$\left(\frac{2y^{3/2}}{x} - \frac{e^x}{x^2}\right)dx + \sqrt{y}\,dy = 0 \; ; \; x>0 \text{ and } y>0 \quad \text{--- } (*)$$

$$\frac{2y}{x} - \frac{e^x}{x^2}\frac{1}{\sqrt{y}} + y' = 0$$

$$y' + \frac{2}{x}y = \frac{e^x}{x^2}y^{-1/2}, \quad n = -\tfrac{1}{2} \quad 1$$

then we have a Bernoulli's D.Eq. Then

$$y'y^{1/2} + \frac{2}{x}y^{3/2} = \frac{e^x}{x^2}, \quad \text{we put } u = y^{3/2}$$

$$u' = \frac{3}{2}y'y^{1/2} \quad \text{or} \quad \frac{2}{3}u' = y'y^{1/2} \quad 1$$

$$\frac{2u'}{3} + \frac{2}{x}u = \frac{e^x}{x^2} \quad \text{(Linear)}$$

$$u' + \frac{3}{x}u = \frac{3}{2}\frac{e^x}{x^2},$$

$$\mu(x) = e^{\int \frac{3}{x}dx} = x^3 \quad 1$$

$$u\mu(x) = x^3 u = \frac{3}{2}\int x^3 \frac{e^x}{x^2}dx = \frac{3}{2}\int x e^x dx$$

$$x^3 u = \frac{3}{2}[xe^x - e^x] + c$$

$$u = \frac{3}{2}\left(\frac{1}{x^2} - \frac{1}{x^3}\right)e^x + c$$

$$\boxed{y^{3/2} = \frac{3}{2}\left(\frac{1}{x^2} - \frac{1}{x^3}\right)e^x + c} \quad 1$$ is the Solution of the D.E (*)

Question 4

$$\frac{dP}{dt} = kP, \quad \frac{dP}{P} = k\,dt$$

$$\ln P(t) = kt + c, \text{ then } \boxed{P(t) = c_1 e^{kt}, \; c_1 \neq 0}$$

Let P_0 be the initial population i.e. $P(0) = P_0$, then

$$P_0 = P(0) = c, \Rightarrow \quad P(t) = P_0 e^{kt}.$$

Now $P(5) = 2P_0$, and $P(10) = 20000$. Then

$$P(5) = 2P_0 = P_0 e^{5k} \Rightarrow 2 = e^{5k} \text{ or } k = \frac{\ln 2}{5}$$

$$P(t) = P_0 e^{\frac{\ln 2}{5} t}$$

$$P(10) = P_0 e^{\left(\frac{\ln 2}{5}\right)10} = 20000$$

$$P(4) = P_0 e^{\ln 4} = 20000 = 4P_0$$

$$\boxed{P_0 = \frac{20000}{4} = 5000}$$