King Saud University.
College of Sciences
Mathematical Department.

Mid-Term1 /S1/2015
Full Mark:25. Time 1H30mn
02/01/1437

Question 1[4,4]. **a)** Find and sketch the largest region of the xy-plane for which the initial value problem

$$\begin{cases} (x^2+y^2)\frac{dy}{dx} = x\sqrt{y-1} \\ y(-2) = 4. \end{cases}$$

has a unique solution.

b) Find the solution of the differential equation:

$$\frac{dy}{dx}(y-1)\sqrt{x^2+1} + x^3 - x = y(x^3+x) \;;\; y \neq 1.$$

Question 2[4,4]. **a)** Solve the following differential equation

$$(3xy - x + y^2) + (x^2 + xy)\frac{dy}{dx} = 0 \;;\; x > 0 \,,\, x + y \neq 0.$$

b) Find the solution of the initial value problem

$$\begin{cases} \left[x\cos^2(\frac{y}{x}) - y\right]dx + xdy = 0 \\ y(1) = \frac{\pi}{4}. \end{cases} \quad : x > 0$$

Question 3[4]. Find the general solution of the differential equation

$$y^3\frac{dy}{dx} + xy^4 = xe^{-x^2} \;;\; x > 0 \,,\, y \neq 0.$$

Question 5[5]. Find the family of orthogonal trajectories for the family of curves

$$y = e^{C\sin x} \;:\; 0 < x < \frac{\pi}{2}\,,$$

where C is an arbitrary constant such that $C \neq 0$.

Complete solutions of M. Term 1

M. 204. First Semester 1436/1437

Question 1

(a) (4)

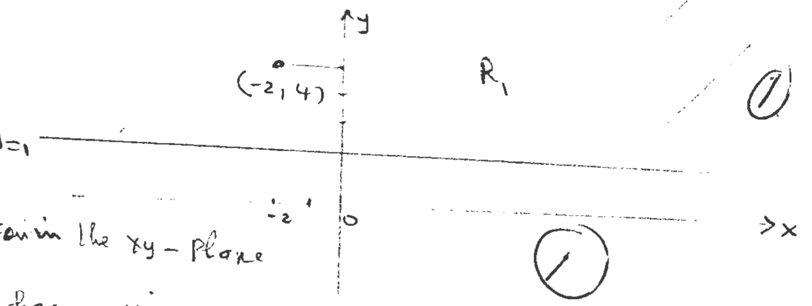
$$y' = \frac{\sqrt{y-1}}{x^2+y^2}\, x = f(x,y) \;;\quad y \geq 1$$

$$y(-2) = 4$$

f is continuous on $R_1 \{(x,y);\ x \in \mathbb{R}\ \&\ y > 1\}$ 1

$$\frac{\partial f}{\partial y} = x\, \frac{\frac{1}{2\sqrt{y-1}}(x^2+y^2) - 2y\sqrt{y-1}}{(x^2+y^2)^2}$$

$\frac{\partial f}{\partial y}$ is also continuous on $R_1 \{(x,y);\ x \in \mathbb{R},\ y > 1\}$. 1



1

So R_1 is the largest region in the xy-plane for which the IVP has a unique solution 1

(b) (4)

$$y'(y-1)(x^2+1)^{1/2} + x(x^2+1) = y(x^2+1)x \;,\quad y \neq 1$$

$$y'(y-1)\sqrt{x^2+1} = x(x^2+1)(y-1)$$

$$y' = x\sqrt{x^2+1}$$ 2

$$dy = x\sqrt{x^2+1}\, dx$$

$$y = \int x(x^2+1)^{1/2} dx = \frac{1}{3}(x^2+1)^{3/2} + c$$ 2

Question 2.

(a) (4)

$$\underbrace{(3xy - x + y^2)}_{M} + \underbrace{(x^2 + xy)}_{N} \frac{dy}{dx} = 0 \quad \text{or} \quad (3xy - x + y^2)dx + (x^2+xy)dy = 0$$

$$\frac{\partial M}{\partial y} = 3x + 2y \;,\quad \frac{\partial N}{\partial x} = 2x + y$$ 1

So the D.E. is not exact. But $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{x+y}{x(x+y)}$, $x(x+y) \neq 0$

$$= \frac{1}{x} = f(x)$$

$$\mu(x) = e^{\int f(x)dx}$$

$$= e^{\int \frac{dx}{x}} = e^{\ln|x|} = |x| = x$$ 1

Then $\mu(x) = x$ is an I. F. of the D.E then $(3x^2y - x^2 + xy^2)dx + (x^3 + x^2y)dy = 0$ is exact equation, hence

$\exists$ $F(x,y)$ s.t

$$\frac{\partial F}{\partial x} = M = 3x^2y - x^2 + xy^2$$

$$\frac{\partial F}{\partial y} = x^3 + x^2y$$

$$F(x,y) = \int (x^3 + x^2y)dy = \boxed{x^3y + \frac{1}{2}x^2y^2 + \phi(x)}$$ (1)

$$\frac{\partial F}{\partial x} = 3x^2y + xy^2 + \phi'(x) = 3x^2y - x^2 + xy^2$$

$$\phi'(x) = -x^2 \Rightarrow \phi(x) = -\frac{1}{3}x^3 + c$$

So the solution of the D.E. is

$$\boxed{F(x,y) = x^3y + \frac{1}{2}x^2y^2 - \frac{1}{3}x^3 + c = 0}$$ (1)

(b) (4)

$$\begin{cases} [x\cos^2(\frac{y}{x}) - y]dx + x\,dy = 0 \\ y(1) = \pi/4 \end{cases} ; x > 0$$

The D.E is homogeneous, we put $\frac{y}{x} = u \Rightarrow y = ux$

$$dy = u\,dx + x\,du$$ (1)

$$[\cos^2(\frac{y}{x}) - \frac{y}{x}]dx + dy = 0$$

$$(\cos^2(u) - u)dx + u\,dx + x\,du = 0$$

$$\cos^2(u)\,dx + x\,du = 0 \Rightarrow \int\frac{dx}{x} + \int\frac{du}{\cos^2(u)} = 0$$ (1)

$$\ln x + \tan u = c \quad \text{or} \quad \boxed{\ln x + \tan(\frac{y}{x}) = c}$$

From $y(1) = \frac{\pi}{4} \Rightarrow \ln 1 + \tan(\pi/4) = 0 + 1 = c$ (1)

Then the solution of the IVP is $\boxed{\ln x + \tan(y/x) = 1}$ (1)

Question 3 (4)

$$y^3 y' + x y^4 = x e^{-x^2} \;, \quad x > 0$$

The D.E is Bernoulli's equation

$$y' + x y = x y^{-3} e^{-x^2} \quad \text{where } n = -3$$

We put $u = y^4$, $u' = 4 y^3 y'$, hence (1)

$$\frac{u'}{4} + x u = x e^{-x^2} \quad \text{or} \quad u' + 4xu = 4x e^{-x^2}$$

$P(x) = 4x$, $\mu(x) = e^{\int P(x)dx} = e^{\int 4x dx} = e^{2x^2}$ (1)

$$\mu(x)\, u = \int 4x e^{-x^2} \cdot e^{2x^2} dx$$

$$e^{2x^2} y^4 = \int 4x e^{x^2} dx = 2 e^{x^2} + c$$ (1)

So the solution of the D.E is

$$e^{2x^2} y^4 - 2 e^{x^2} = c \quad \text{or}$$

$$\boxed{e^{x^2}(y^4 e^{x^2} - 2) = c}$$ (1)

Question 4 (5)

$$y = e^{c \sin x} \quad c \neq 0 \;, \quad 0 < x < \pi/2$$

$\ln y = c \sin x$, $c = \dfrac{\ln y}{\sin x}$ (1)

$$0 = \frac{\frac{y'}{y}\sin x - \cos x \ln y}{(\sin x)^2} \Rightarrow \frac{y'}{y}\sin x - \cos x \ln y = 0$$

$$y' \sin x = y \cos x \ln y$$

$$\boxed{y' = f(x,y) = y \cot x \cdot \ln y}$$ (1)

Now we have to solve the D.E $y' = \dfrac{-1}{f(x,y)} = \dfrac{-1}{y \cot x \ln y} = \dfrac{-\tan x}{y \ln y}$

$$\frac{dy}{dx} = \frac{-\tan x}{y \ln y}$$

$$\int y \ln y \, dy + \int \frac{\sin x}{\cos x} dx = 0$$

$$\boxed{\frac{y^2}{2}\ln y - \frac{1}{4}y^2 - \ln \cos x = c}$$ (1) or

$$\boxed{2y^2 \ln y - y^2 - 4 \ln \cos x = c_1}$$ (2)