King Saud University,
College of Sciences
Mathematical Department.

Mid-Term1(Math 20[illegible]4)
Full Mark:25. Time 1H.30 mn.
26 /12/1434

Question 1.[4,4]. **a)** Find and sketch the largest region of the $xy-$plane for which the initial value problem

$$\begin{cases} \sqrt{x^2-4}\cdot\frac{dy}{dx} = 1 + e^x \ln y \\ y(-3) = 4 \end{cases}$$

has a unique solution.

b) Solve the initial value problem

$$\begin{cases} \frac{dy}{dx} = x - xy - y + 1, \qquad y \neq 1. \\ y(0) = 0. \end{cases}$$

Question 2.[4,4]. **a)** Find the general solution of the differential equation

$$\frac{dy}{dx} - (y - 2x)^2 = 3.$$

b) Solve the differential equation

$$\frac{x^2 y'}{y^4} + \frac{x}{y^3} = \sin x, \qquad x > 0.$$

Question 3.[5]. Initially there were 80 gram of a radioactive material present. After 10 hours the mass decreases by 5%. If the rate of decay is proportional to the amount of the material at time t, then find the half life of this material.

Question 4.[4]. Show that the following differential equation is homogeneous and solve it.

$$(x^2 + y^2)dx + (x^2 - xy)dy = 0.$$

Complete solutions of First Mid-Term

Math. 204. First Sem. 1434-35H

Question 1.

(a) IVP: $\begin{cases} \sqrt{x^2-4}\,\dfrac{dy}{dx} = 1 + e^x \ln y \\ y(-3) = 4 \end{cases}$

$$\frac{dy}{dx} = y' = \frac{1}{\sqrt{x^2-4}} + \frac{e^x}{\sqrt{x^2-4}} \cdot \ln y \ ; \quad y > 0 \text{ and } |x| > 2 \qquad (1)$$

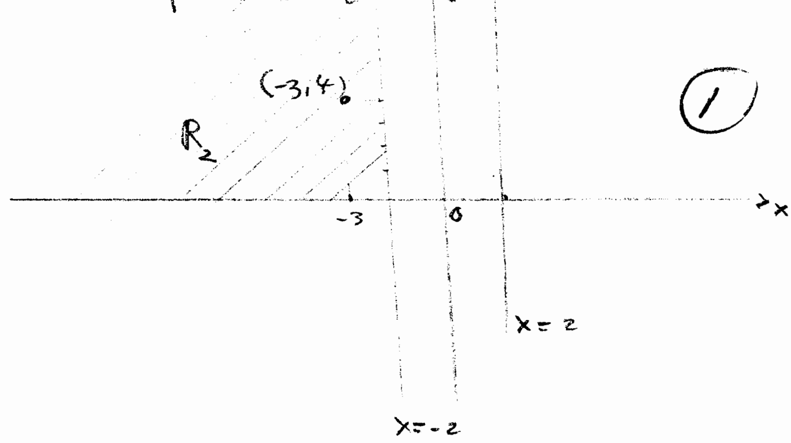
$$y' = f(x,y)$$

$$\frac{\partial f}{\partial y} = \frac{e^x}{\sqrt{x^2-4}} \cdot \frac{1}{y}$$

f and $\frac{\partial f}{\partial y}$ are continuous on $R = \{(x,y) ; |x| > 2 \text{ and } y > 0\}$ (2)

$$R = \underbrace{\{(x,y) ; x > 2 \ \& \ y > 0\}}_{R_1} \cup \underbrace{\{(x,y) ; x < -2 \ \& \ y > 0\}}_{R_2}$$

As $(-3, 4) \in R_2$, then R_2 is the largest region for which the IVP has a unique solution



(1)

(4) (b)

$$\frac{dy}{dx} = x - xy - y + 1 = x(1-y) + (1-y)$$

$$y' = (x+1)(1-y) \qquad (2)$$

$$\frac{dy}{1-y} = (x+1)dx \Rightarrow -\ln|1-y| = \frac{1}{2}(x+1)^2 + c$$

At $x=0$, $y=0 \Rightarrow -\ln|1-0| = \frac{1}{2} + c = 0$ (2)

hence $c = -\frac{1}{2}$, then the solution of the IVP is

$$\boxed{\frac{1}{2}(x+1)^2 + \ln|1-y| = \frac{1}{2}} \quad \text{or} \quad \boxed{(x+1)^2 + \ln(1-y)^2 = 1}$$

Question 2

(a) $$\frac{dy}{dx} - (y-2x)^2 = 3$$

Let $u = y - 2x$, $y' = u' + 2$, $\Rightarrow u' + 2 - u^2 = 3$ 2

$$u' = u^2 + 1 \Rightarrow \frac{du}{u^2+1} = dx \Rightarrow \tan^{-1}(u) = x + c.$$

Then the solution of the D.E is 2

$$\boxed{\tan^{-1}(y-2x) - x = c}$$

(b) $$\frac{x^2 y'}{y^4} - \frac{x}{y^3} = \sin x; \quad x>0 \text{ and } y \neq 0.$$ is Bernoulli's eq.

then $x^2 y' - xy = y^4 \sin x$ or $y' - \frac{1}{x} y = \frac{\sin x}{x^2} y^4$; $n = 4$.

hence $$y' y^{-4} - \frac{1}{x} y^{-3} = \frac{\sin x}{x^2}; \quad x>0 \ \& \ y \neq 0$$ 2

We put $w = y^{-3} \Rightarrow w' = -3 y^{-4} y'$, so $y^{-4} y' = -\frac{w'}{3}$

and $$-\frac{w'}{3} - \frac{1}{x} w = \frac{\sin x}{x^2}$$

or $$\boxed{w' + \frac{3}{x} w = -3 \frac{\sin x}{x^2}}$$ which is a first order linear D.E.

$$\mu(x) = e^{\int \frac{3}{x} dx} = e^{\ln x^3} = x^3$$ 1

then $$w x^3 = \int -3 \left(\frac{\sin x}{x^2}\right) x^3 \, dx$$

$$= -3 \int x \sin x \, dx = -3(-x\cos x + \sin x) + C$$

or $$y^{-3} x^3 = 3x\cos x - 3\sin x + C_1 \qquad (C_1 = -3C)$$ 1

Hence the solution of the D.E is given by the family of curves

$$\boxed{x^3 = (3x\cos x - 3\sin x + C_1) y^3}$$

Question ③ $\frac{dm}{m} = k\,dt \Rightarrow \ln m = kt + c$, here $m > 0$

or $m(t) = e^{kt} \cdot e^{c}$

$\boxed{m(t) = c_1 e^{kt}}$; $c_1 = e^{c} > 0$

(1)

At $t = 0$, $m(0) = 80$ grams $= m_0$

At $t = 10$ $m(10) = 80 - 80\left(\frac{5}{100}\right) = 80 - 4 = 76$ grams.

then $m(0) = \boxed{80 = c_1}$, $m(t) = 80\,e^{kt}$ (1)

$m(10) = 80\,e^{10k} = 76$ or $\frac{76}{80} = e^{10k}$

hence $k = \frac{1}{10}\ln\left(\frac{76}{80}\right) \Rightarrow \boxed{k \simeq -0.005}$, $\boxed{m(t) = 80\,e^{-0.005t}}$

(1)

Now if $m(t) = \frac{m_0}{2} = 40$ grams, then

$40 = 80\,e^{-0.005t} \Rightarrow \frac{1}{2} = e^{-0.005t}$ (2)

or $t = \frac{-\ln(2)}{-0.005} \Rightarrow \boxed{t \simeq 138.63 \text{ hours}}$

Question ④ $(x^2+y^2)dx + (x^2-xy)dy = 0$ is homogeneous

We suppose $x > 0$, and put $u = \frac{y}{x}$, $y = xu \Rightarrow dy = x\,du + u\,dx$

$$\left[1 + \left(\frac{y}{x}\right)^2\right] dx + \left[1 - \left(\frac{y}{x}\right)\right] dy = 0$$

$$(1+u^2)dx + (1-u)(x\,du + u\,dx) = 0$$

$$(1 + \cancel{u^2} + u - \cancel{u^2})dx + (1-u)x\,du = 0$$

$$(1+u)\,dx + x(1-u)\,du = 0$$

(2)

$$\frac{dx}{x} + \left(\frac{1-u}{1+u}\right)du = 0; \quad u \neq -1; \ (y \neq -x)$$

$$\frac{dx}{x} + \left(-1 + \frac{2}{u+1}\right)du = 0 \Rightarrow$$

(2)

$$\ln x + 2\ln|u+1| - u = c \text{ or}$$

$\boxed{\ln x + \ln\left(1 + \frac{y}{x}\right)^2 - \frac{y}{x} = c}$ is the solution of the D.E.