

Question 1[4]. Find and sketch the largest local region of the xy -plane for which the initial value problem

$$\begin{cases} \sqrt{2 - \ln(y - 3)}dx - (x^2 - 5x - 6)dy = 0 \\ y(5) = 4, \end{cases}$$

has a unique solution.

Question 2[4+4]. a) Solve the initial value problem

$$\begin{cases} x(1 + y)dx + (x^2 - 1 + y^2)dy = 0, & y > -1, x \neq 0, \\ y(1) = 1. \end{cases}$$

b) Reduce the following equation to a Bernoulli equation and obtain its general solution

$$(1 + x^2)^2 y \frac{dy}{dx} + 2x(1 + x^2)y^2 - 1 = 0, \quad y \neq 0.$$

Question 3[4+4]. a) Solve the differential equation

$$\frac{dy}{dx} - e^{-y} \cos(2x) - 2 = -2 \cos^2(x) - e^{-y}, \quad y \neq 0.$$

b) Find the general solution of the differential equation

$$(2x + y + x \ln x)dx - 2x dy = 0, \quad x > 0.$$

Question 4[5]. The sum of 5000 SAR is invested at a rate of 8% per year. Compounded contiguously. What will be the amount after 25 years. *

Question 1

$$\begin{cases} y' = \frac{dy}{dx} = \frac{[2 - \ln(y-3)]^{1/2}}{x^2 - 5x - 6} = f(x,y) \\ y(5) = 4 \end{cases}$$

f is continuous on $R_1 = \{(x,y) : x \neq 6, x \neq -1, 2 - \ln(y-3) > 0 \Rightarrow y > 3\}$

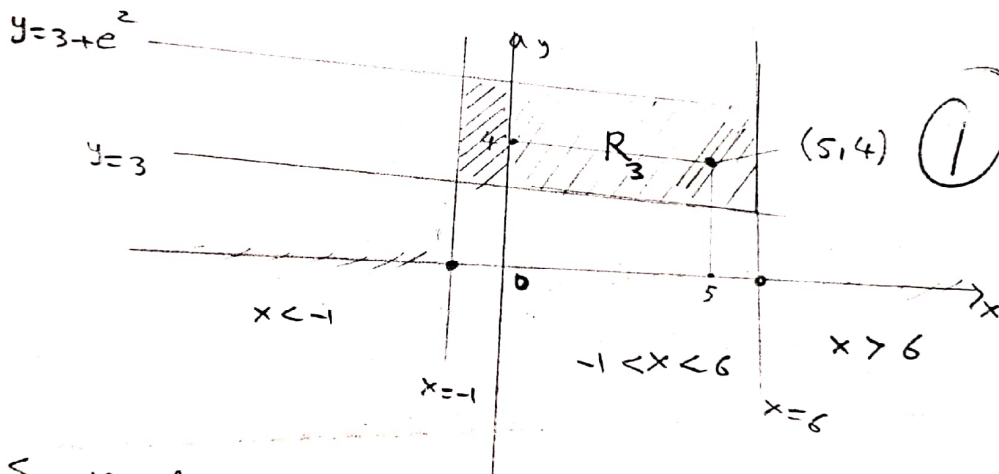
$$\frac{\partial f}{\partial y} = \frac{1}{(x-6)(x+1)} \left(\frac{1}{2}\right) (2 - \ln(y-3))^{-1/2} \frac{(-1)}{y-3}$$

then f and $\frac{\partial f}{\partial y}$ are continuous on

$$R_2 = \{(x,y) : x \neq 6, x \neq -1, 2 - \ln(y-3) > 0 \text{ and } y > 3\}$$

So $\ln(y-3) < 2 \Rightarrow y-3 < e^2$ or $y < 3 + e^2$

Hence $3 < y < 3 + e^2$, but $5 \in (-1, 6)$



So the largest local region of xy -plane for which the IVP has a unique solution is

$$R_3 = \{(x,y) : -1 < x < 6, 3 < y < 3 + e^2\}$$

Question 2

(a) $\begin{cases} x(1+y)dx + (x^2-1+y^2)dy = 0, & x \neq 0, y > -1 \\ y(1) = 1 \end{cases}$

$M = x(1+y), N = x^2-1+y^2$
 $\frac{\partial M}{\partial y} = x, \frac{\partial N}{\partial x} = 2x$
 $\frac{\frac{\partial M}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{x}{x(1+y)} = \frac{1}{1+y}$
 $\mu(y) = e^{\int \frac{dy}{y+1}} = e^{\ln|y+1|} = y+1$ (1)

Hence

$x(1+y)^2 dx + (x^2-1+y^2)(y+1)dy = 0$ is exact D.E.

Then there exists $F(x,y)$ s.t

$\frac{\partial F}{\partial x} = x(1+y)^2, \frac{\partial F}{\partial y} = (x^2-1+y^2)(y+1)$

So $F(x,y) = \int x(1+y)^2 dx = \frac{1}{2}x^2(1+y)^2 + \phi(y)$

$\frac{\partial F}{\partial y} = \phi'(y) + x^2(1+y) = x^2(y+1) - (y+1) + y^2(y+1)$

$\phi'(y) = y^3 + y^2 - y - 1$

$\phi(y) = \frac{1}{4}y^4 + \frac{1}{3}y^3 - \frac{1}{2}y^2 - y + C$

Conclusion $F(x,y) = \frac{1}{2}x^2(1+y)^2 + \frac{1}{4}y^4 + \frac{1}{3}y^3 - \frac{1}{2}y^2 - y + C = 0$

is the solution of the D.E.

But $y(1) = 1 \Rightarrow \frac{1}{2}(4) + \frac{1}{4} + \frac{1}{3} - \frac{1}{2} - 1 + C = 0$

$C = -\frac{13}{12}$ (1)

$\frac{1}{2}x^2(1+y)^2 + \frac{1}{4}y^4 + \frac{1}{3}y^3 - \frac{1}{2}y^2 - y - \frac{13}{12} = 0$ is the solution of the I.V.P

(b) $(1+x^2)^2 y \frac{dy}{dx} + 2x(1+x^2)y^2 - 1 = 0, y \neq 0$

$\frac{dy}{dx} + \frac{2x}{(1+x^2)}y = \frac{1}{(1+x^2)^2}y^{-1}$ is B. D. Eq, $n = -1$

Then

$y y' + \frac{2x}{(1+x^2)}y^2 = \frac{1}{(1+x^2)^2}$ (2)

We put $u = y^2$, $u' = 2yy'$, $yy' = \frac{u'}{2}$

$u' + \frac{4x}{(1+x^2)}u = \frac{2}{(1+x^2)^2}$ is linear D.E

$\mu(x) = e^{\int \frac{4x dx}{1+x^2}} = e^{2 \ln(1+x^2)} = (1+x^2)^2$ (1)

$u \mu(x) = u(1+x^2)^2 = \int \frac{2}{(1+x^2)^2} \cdot (1+x^2)^2 dx = \int 2 dx$

$u(1+x^2)^2 = 2x + c$, then

$y^2(1+x^2)^2 = 2x + c$ is the solution of the D.E. (1)

Question 3

(a) $y' - e^{-y} \cos(2x) - 2 = -2 \cos^2(x) - e^{-y}$; $y \neq 0$

$y' - e^{-y} \cos(2x) - 2 = -(1 + \cos(2x)) - e^{-y}$ (1)

$y' - e^{-y} \cos(2x) + \cos(2x) = 1 - e^{-y}$

$y' + \cos(2x)(1 - e^{-y}) = (1 - e^{-y})$

$\frac{dy}{1 - e^{-y}} + (\cos(2x) - 1) dx = 0$ (1)

$\int \frac{e^y dy}{e^y - 1} + \int (\cos(2x) - 1) dx = 0$ (2)

So $\ln|e^y - 1| + \frac{1}{2} \sin(2x) - x = c$ is the solution of the D.E.

(b) $(2x + y + x \ln x)' dx - 2x dy = 0$; $x > 0$

$y' - \frac{1}{2x}y = \frac{2 + \ln x}{2} = 1 + \frac{1}{2} \ln x$ (1)
is Linear D.E.

$$\mu(x) = e^{-\int \frac{dx}{2x}} = e^{-\frac{1}{2} \ln x} = e^{\ln x^{-1/2}} = \frac{1}{\sqrt{x}} \quad (1)$$

$$\begin{aligned} y \mu(x) &= \int \frac{1}{\sqrt{x}} \left(1 + \frac{1}{2} \ln x\right) dx \\ &= \int \frac{1}{\sqrt{x}} dx + \frac{1}{2} \int x^{-1/2} \ln x dx \\ &= 2\sqrt{x} + \frac{1}{2} [2x^{1/2} \ln x - 4x^{1/2}] + C \end{aligned}$$

$$\frac{y}{\sqrt{x}} = 2\sqrt{x} + x^{1/2} [\ln x - 2] + C \quad (2)$$

$$y = 2x + x(\ln x - 2) + \sqrt{x} C$$

$y = x \ln x + \sqrt{x} C$ is the Gen. solution of the D.E

Question 4

Let $y(t)$ be the amount of money at time t .

$$\frac{dy}{dt} = r = \frac{8}{100}, \quad \frac{dy}{y} = \frac{8}{100} dt \quad (3)$$

$$\text{Then } y(t) = C e^{(\frac{8}{100})t} \quad (4)$$

$$\text{But } y(0) = 5000, \Rightarrow y(t) = 5000 e^{(\frac{8}{100})t}$$

$$y(25) = 5000 e^{(\frac{8}{100})(25)} = 5000 e^2 \approx 36,945,28 \text{ SAR} \quad (5)$$